REPRESENTATION OF FUNCTIONALS BY INTEGRALS

By Richard Arens

Let X be a topological space, and let $C_0(X, R)[C_\infty(X, R)]$ be the class of continuous real-valued functions for which the closure of the set on which $|f(x)| > c$, $c > 0$ $[c = 0]$ is compact. Let Q be a subalgebra of $C_0(X, R)$ in which every function can be represented as the difference of non-negative members. Let J be a non-negative linear functional defined in Q such that

$$J(f) = \sup J(fg) \qquad (0 \leq g \leq 1,\ g \in Q). \tag{1.1}$$

Our main theorem says that then there exists a complete *strongly regular* (see §3) measure m in X such that each f in Q is measurable and

$$J(f) = \int f(x)\ m(dx), \tag{1.2}$$

and of which all other complete measures satisfying (1.2) are extensions.

Since (1.1) is clearly satisfied when $Q = C_\infty(X, R)$, this result includes the theorems of Riesz [7], Saks [8], Markoff [6], Kakutani [4]. Of course, it does not yield anything more than these theorems in the cases to which they apply. However, there are cases in which none of these results can be immediately applied, in harmonic analysis (see [5]). Our procedure is to extend the functional J by means of a lemma generalizing a device of Krein's, to a class Q_2, containing a sufficiently large subclass Q_1 of functions to which Daniell's general integral theorem [2] can be applied. Finally (1.1) is used to show that the integral representation valid for Q_1 is valid even for Q_2 (and hence Q).

2. The functions considered in this section are defined on spaces whose topology, if any, does not enter the picture. The problem is to extend a non-negative linear functional from a space Q of functions to a space Q_2 which contains "densely" (see Theorem 2.3) Q and another space Q_1 which is "functionally complete", which notion we now introduce.

Let C be a class of real valued functions defined on any domain X. We shall say that C is *functionally complete* if, given a real valued continuous function φ defined in real n-dimensional space such that $\varphi(0, 0, \cdots, 0) = 0$, and n members $f, g, \cdots, h$ of C, the function $\varphi(f, g, \cdots, h)$ with values $\varphi(f(x), g(x), \cdots, h(x))$ also belongs to C.

A functionally complete class of functions is thus a linear algebra and a vector lattice.

The first theorem shows that many functionally complete spaces exist and are easily obtainable.

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