THE EXISTENCE OF PERIODIC SOLUTIONS OF SYSTEMS OF DIFFERENTIAL EQUATIONS

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Throughout this paper we will consider systems of differential equations of the form

$$(1) \qquad \dot{x} = X(x, t)$$

where x represents a vector in the m-dimensional space, $\dot{x}$, its derivative with respect to the time t, and $X(x, t)$ is a sufficiently smooth vector function, defined in the whole (x, t) space, so that the standard existence and uniqueness theorems hold. We will assume, moreover, that X is periodic in t of period 1.

It is well known that such systems may have a wide variety of periodic solutions (often called, in the sequel, vibrations or oscillations). The period of these solutions may be 1, *i. e.*, the same as the period of the system, or any integer q, or any rational or even, in some exceptional cases, irrational number. The family of all the periodic solutions is, however, far from being arbitrary, because certain relations between the number and types of solutions of the different periods must be satisfied.

We will call *harmonic vibration or oscillation* any periodic solution which admits the period 1; in particular, the constant solutions and the "superharmonic" solutions, of period $1/q$, q integer, are considered here as harmonic. We will call *subharmonic vibration or oscillation of order* q (q integer) any periodic solution which admits the period q and no smaller *integral* period; in particular, any solution of rational period which is not a "superharmonic" is considered here as a subharmonic.

The first question that naturally arises is the following: does the existence of periodic solutions (of any period) imply the existence of at least one harmonic vibration? The answer, which is negative in general but affirmative under some mild assumptions, if the order of the system is not higher than 2, will be discussed in the first part of this paper.

In the second part, we discuss the nature of the family of the subharmonic vibrations and show that, except for the restrictions which derive from the theorems of Part I, this family may be quite arbitrary.

I. We will say that a certain property holds "in the future" if it is true for any value of t greater than a certain finite value t_0. The following simple existence theorem may then be proved.

THEOREM 1. *If the system* (1) *is of the first order* ($m = 1$), *the existence of a solution which exists and remains bounded in the future implies the existence of a harmonic vibration.*

Received January 3, 1949.