A THEOREM OF CARTAN

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Consider a non-degenerate linear complex K in the projective space of three dimensions. A curve C is said to be a curve of the complex if at each of its points M it is tangent to the focal plane P of the complex at M. It is known that such a curve C has at each of its points M contact of the second order with the focal plane P of K at M. It may therefore be said that all of the curves of the complex are asymptotic lines of the complex. Cartan [1] investigated the question of the possible existence of a reciprocal of this theorem. He was able to prove such a theorem by solving the following general problem: In a real projective space of r dimensions, $r \geq 3$, let a distinct hyperplane P be attached according to a determined law to each point M in the space. A field of contact elements (M, P) is thus defined in the space. A curve of the field is defined to be a curve which is tangent at each of its points M to the hyperplane P associated with this point. Determine the fields which possess the property that every curve of the field has at each of its points M contact of the second order with the hyperplane associated with this point. Cartan calls such a field of hyperplanes a field (C). He finds these fields to be of two classes, (1) and (2), according as the equation of Pfaff which defines the field (C) in a domain D of the space is completely integrable or is not completely integrable, respectively. The first class consists of fields which are confined to local domains D of the space, whereas the fields of the second class constitute "global" solutions of the problem which extend to all points of the space. The reciprocal theorem which Cartan proved states that *if the equation of Pfaff which defines the curves of a field* (C) *is not completely integrable, the hyperplane* P *associated with a point* M *is the focal hyperplane of the point* M *with respect to a fixed linear complex* [1; 216–219]. The purpose of this note is to determine the fields (C) which constitute the "global" solutions of the problem in a very simple manner and thereby obtain an elementary proof of Cartan's theorem.

Let $x^0, x^1, \cdots, x^r$ and $\xi_0, \xi_1, \cdots, \xi_r$ denote projective homogeneous coordinates of a point M and hyperplane P, respectively. If the point M lies on the hyperplane P, the coordinates of M and P satisfy the equation

$$\xi_i x^i = 0 \tag{1}$$

in which the usual summation convention of tensor analysis is observed. A field F of contact elements (M, P) is defined by (1) if the functions are continuous single valued functions of the $r + 1$ coordinates of M. Let a curve C be defined by $r + 1$ coordinates of M as functions of a parameter t. The curve

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