A PORISTIC SYSTEM OF TRIANGLES

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The subject of poristic systems of triangles has engaged the attention of geometers from the time of Poncelet; in fact, they are sometimes referred to as Poncelet triangles. Studies of loci of notable points associated with triangles of such a system were made by Laguerre in 1879 [5] and by Weill in 1880 [11]. More recently Goormaghtigh [4] using isotropic coordinates proved some of the results Laguerre and Weill had obtained by synthetic means. In the present paper the author also employs isotropic coordinates to obtain the principal results obtained by these authors. The technique employed, however, is believed to be new and more simple and direct.

Let the base circle be the unit circle in a system of isotropic coordinates. Let C and C' represent two triangles inscribed in this circle. We represent the vector coordinates of the vertices of C and C' by t_1, t_2, t_3, t_1', t_2', t_3', respectively. Then, as is well known, the sides of the triangles touch a conic Γ whose foci are points isogonally conjugate with respect to the triangles C and C'. Moreover C and C' are only two triangles of an infinite set of triangles that can be inscribed to the circle and which are at the same time circumscribed to the same conic. A system of triangles inscribed to one curve and at the same time circumscribed to a second curve is said to be poristic.

1. A cubic involution. The vertices of triangle C can be collectively represented by the polynomial $C \equiv (z - t_1)(z - t_2)(z - t_3) \equiv z^3 - \sigma_1 z^2 + \sigma_2 z - \sigma_3$; similarly for C'. Consider the pencil $C_\lambda \equiv C + \lambda C'$ where λ is a parameter, not necessarily real. The zeros of C_λ will represent points on the unit circle, including possibly pairs of points inverse to the circle when [8; 508] $|\sigma_1 + \lambda\sigma_1'| = |\sigma_2 + \lambda\sigma_2'|$, $|\sigma_3 + \lambda\sigma_3'| = |1 + \lambda|$,

$$\operatorname{amp}(\sigma_1 + \lambda\sigma_1')(1 + \lambda)^{-1}(\sigma_2 + \lambda\sigma_2')(1 + \lambda)^{-1} = \operatorname{amp}(\sigma_3 + \lambda\sigma_3')(1 + \lambda)^{-1}.$$

These conditions are simultaneously satisfied when $\lambda/\lambda^* = \sigma_3/\sigma_3'$, that is, when $\lambda = kt^2$, where k is a real variable, and $t^2 = e^{2i\theta}$, a constant, since σ_3 and σ_3' are constants. The angle θ is the angle referred to by Goormaghtigh [3; 201] as the "associated" angle of the two triangles C and C'. The associated angle is simply the angle between the Simson lines of any point on the circumcircle with respect to the two triangles, and it is constant [1; 218]. Hence for any value of λ for which $\lambda/\lambda^* = \sigma_3/\sigma_3'$ we obtain a triangle C_λ of the poristic system determined in C and C'. It may happen that the conic Γ intersects the base circle in two or four real points. In this case for such points as lie on the circle

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