# THE STRUCTURE OF THE REGULAR REPRESENTATION OF CERTAIN DISCRETE GROUPS

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Let $G$ be a locally compact topological group with a countable base for its open sets. Let $L_2(G)$ be the space of those complex valued Haar measurable functions $x(g)$ on $G$ for which $\int | x(g) |^2 dg$ is finite, where $dg$ refers to left-invariant Haar measure on $G$. Let us consider the "regular representation" of $G$: With every group element $h$ we associate the operator $U(h)$ acting in $L_2(G)$ and defined by $U(h)$: $x(g) \rightarrow x(h^{-1}g)$. By $W$ we denote the weakly closed self-adjoint algebra (that is, a "ring of operators" in the sense of [5]) of bounded linear operators generated by the operators $U(h)$. According to a result of von Neumann [7; Theorem VII] one can decompose the Hilbert space $L_2(G) = H$ into a "generalized direct sum" or "direct integral" $\int_+ H_t$ of Hilbert (or finite dimensional) spaces $H_t$ under the operators $A \varepsilon W$ such that almost all the operator algebras $W(t)$ (acting in $H_t$) are factors in the sense of [5] (that is, weakly closed self-adjoint operator algebras whose centers consist of the scalar multiples of the identity operator). {Given a family of Hilbert spaces or finite dimensional vector spaces $H_t$ over the complex numbers and a measure space of elements $t$; a Hilbert space $H$ is the direct integral $\int_+ H_t$ of the spaces $H_t$ if every $x$ of $H$ can be represented by a vector valued function $x(t)$ with values in $H_t$ such that $x(t)$ is "summable" in a certain sense; and conversely to every such $x(t)$ there is an $x \varepsilon H$. (See [7] for the precise definition.) The measure space can be taken to be the real line and the measure a Lebesgue-Stieltjes measure. When we speak about "almost every $t$" (or "almost every $H_t$ or $W(t)$") we mean of course all $t$ (all $H_t$ or $W(t)$) except for a set whose measure (with which the given $\int_+ H_t$ is formed) is zero.} The main problem in this connection is: *What are the properties of the group G which determine what kind of factors one gets in this decomposition?* For instance if $G$ is compact or commutative or a direct product of compact and commutative groups, then it is well known that almost all the spaces $H_t$ must be finite dimensional; hence, by the structure theorems of Wedderburn, almost each $W(t)$ must be a finite-dimensional simple algebra over the complex numbers, isomorphic to some total matrix algebra. And it is essentially this fact which allows one to conclude that the Peter-Weyl completeness relation [8] and the Plancherel theorem express the square integral $\int | x(g) |^2 dg$ in terms of the ordinary trace defined on the finite dimensional matrix algebras $W(t)$.

In the general case one still obtains such an expression for $\int | x(g) |^2 dg$ in terms of the relative traces which Murray and von Neumann have shown to exist on certain factors (see [5]). Thus the precise nature of this "completeness

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