WEAK CONVERGENCE IN THE SPACES H^p

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1. **Introduction.** By H^p we mean the class of functions f which are analytic for $| z | < 1$ and such that

$$\mathfrak{M}_p(f; r) = \left(\frac{1}{2\pi} \int_0^{2\pi} | f(re^{i\theta}) |^p \, d\theta\right)^{1/p}$$

is bounded when $0 \leq r < 1$. This class has interesting properties for any $p > 0$. We shall assume $p \geq 1$ however, for we are interested in H^p as a complex Banach space, the norm being defined as $|| f ||_p = \sup \mathfrak{M}_p(f; r)$, the supremum being taken with respect to r over the range $0 \leq r < 1$. It is not difficult to prove that $|| f ||_p$ has the properties of a norm, and that with this norm H^p is a complete space. When $p = 1$ we omit the index, writing H for H^1 and $|| f ||$ for $|| f ||_1$.

In this paper our entire concern is with the question of weak convergence in H^p. Our investigations are not based on the representation of the linear functionals on H^p (we have studied these representations elsewhere, with results as yet unpublished), but on the fact that there is a natural isometric imbedding of H^p in L^p. The problem then becomes one of discovering how the known conditions for weak convergence in L^p can be translated into conditions of interest in H^p.

The results for H are stated in Theorem 5.2, and those for H^p $(p > 1)$ in Theorem 6.1. The genuine difference between the cases $p = 1$ and $p > 1$ is pointed out by an example at the end of §5. Theorems 5.2, 6.1, and the example just mentioned are the main results of the paper. In §7 we use the example of §5 to show that the unit sphere of H is not weakly sequentially compact, and hence that H is not reflexive.

The bulk of the analysis is in §§4 and 5; the arguments are arranged in two series of lemmas leading to Theorems 4.1 and 5.1 respectively. In all this work I was much influenced by reading the important memoir of Fichtenholz [3].

2. **The imbedding of H^p in L^p.** We shall need the following facts, which are well known:

(a) $\mathfrak{M}_p(f; r)$ is a nondecreasing function of r.

(b) If $f \in H^p$, $\lim_{r \to 1} f(re^{i\theta}) = f(e^{i\theta})$ exists for almost all θ; $f(e^{i\theta}) \in L^p(0, 2\pi)$ and

$$|| f ||_p = \left(\frac{1}{2\pi} \int_0^{2\pi} | f(e^{i\theta}) |^p \, d\theta\right)^{1/p}$$

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