CREMONA'S EQUATIONS AND THE PROPERNESS INEQUALITIES

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1. **Introduction.** Let a regular linear system of plane curves of dimension r be determined by prescribing its order x_0 and its multiplicities $x_1, x_2, \cdots, x_n$ at a set of base points $P_1, P_2, \cdots, P_n$. If the general curve is irreducible, of genus g, and has no singularities except those prescribed at the base points, then the characteristic $x = (x_0, x_1, x_2, \cdots, x_n)$ satisfies *Cremona's equations* [2]:

$$\text{(I)} \qquad \begin{aligned} x_0^2 - x_1^2 - x_2^2 - \cdots - x_n^2 &= r + g - 1, \\ 3x_0 - x_1 - x_2 - \cdots - x_n &= r - g + 1. \end{aligned}$$

On the other hand, there exist solutions which are not characteristics of such linear systems (Ruffini [8; 5], [9; 483] erroneously gave $(10; 6, 4, 3^5, 1^2)$ as describing a homaloidal net). Those solutions of (I) which are characteristics of such linear systems are said to be *proper*. For reasons given below, the $(n + 1)$-tuples of type $(0; 0, \cdots, 0, -1, 0, \cdots, 0)$ are defined to be proper.

If x is a proper solution and p is a proper solution of (I) for $g = r = 0$ and such that $p_0 < x_0$, then [7]

$$\text{(II)} \qquad p_0x_0 - p_1x_1 - p_2x_2 - \cdots - p_nx_n \geq 0 \qquad (x_0 > p_0).$$

For $p_0 > 0$, this asserts that the total intersection multiplicity of the curve of characteristic p with an irreducible curve of the system of characteristic x can not exceed p_0x_0. For p of the type $(0; 0, \cdots, 0, -1, 0, \cdots, 0)$, this states that the multiplicities in a characteristic x with $x_0 > 0$ are non-negative. For any x, proper or not, there is defined a finite set of inequalities (II), the *properness inequalities* for that x It has been conjectured [3; 11, 15] that solutions of Cremona's equations which also satisfy the properness inequalities are proper. This paper examines the arithmetic implications of (I) and (II) and shows in particular that the answer is in the affirmative for $g = 0, 1$ and $r > 0$.

For any n, the set of all x for which x_0, r, and g are non-negative will be designated by A_n. In particular, A_n contains all proper solutions. The bilinear form $x_0y_0 - x_1y_1 - \cdots - x_ny_n$ is abbreviated to (xy). Certain $(n + 1)$-tuples are represented by a, b_i, c, and d_{ijk} : $b_i = (0; 0, \cdots, 0, -1, 0, \cdots, 0)$, where $x_i = -1$ and $x_s = 0$ for $s \neq i$; $c = (1; 0, 0, \cdots, 0)$; $a = 3c - \sum_i b_i$; and $d_{ijk} = c - b_i - b_j - b_k$. In this notation, the fundamental relations take the form:

$$\text{(I)} \qquad (xx) = r + g - 1, \qquad (ax) = r - g + 1,$$

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