VARIOUS TYPES OF ORTHOGONALIZATION

By Menahem Schiffer

Introduction. The method of orthogonal functions has been applied successfully to the theory of analytic functions of a complex variable and to various problems of partial differential equations of elliptic type [2]–[9], [17]–[20]. The value of orthonormal functions for solving boundary value problems and constructing fundamental solutions has been stressed. Various types of orthogonalization have been proposed and applied successfully. We mention as an example that in the theory of functions which are analytic in a given domain B with boundary C the following two orthogonal systems have been used: The set of functions $\{f_\nu(z)\}$ analytic in B and satisfying the conditions

$$\oint_C f_\nu(z) f_\mu^-(z)\, ds = \delta_{\nu\mu}\,, \tag{1}$$

which has been studied first systematically by Szegö [17], [18], and the set $\{g_\nu(z)\}$ which is orthogonalized by the requirements

$$\iint_B g_\nu(z) g_\mu^-(z)\, d\tau = \delta_{\nu\mu}\,; \tag{2}$$

the latter type of orthogonalization has been introduced by Bergman [2] and Bochner [8].

Remark. In this paper x^-, $f^-(x)$, $(T)^-$ denote the complex numbers conjugate to x, $f(x)$, T respectively.

The different theories of orthonormal function systems $\{f_\nu(z)\}$ have a great similarity since their formal background is identical; in fact, we deal in any case with abstract linear spaces upon which a certain metric has been superimposed. It can be shown from this formal point of view that one of the most useful concepts in such a theory is the kernel of the system

$$K(z, \zeta^-) = \sum_{\nu=1}^{\infty} f_\nu(z) f_\nu^-(\zeta) \tag{3}$$

and various properties of this kernel may be established in a very general way. If we ask, however, for the significance of the kernel considered with respect to the domain, the type of the basic system becomes very important and quite different answers are obtained for different orthogonalization.

In the case of the orthogonalizations (1) and (2) the kernels could be identified with interesting and important domain functions which play an independent

Received October 18, 1948. Paper done under contract with the Office of Naval Research.