FUNCTIONS OF BOUNDED VARIATION

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Let $f(x)$ be a function of bounded variation in the interval (a, b); let $I = (a = x_0 < x_1 < \cdots < x_n = b)$, an arbitrary subdivision of the interval (a, b); and let $S(I) = \sum_{i=1}^{n} | f(x_i) - f(x_{i-1}) |$. Then the total variation V of $f(x)$ in the interval (a,b) is defined by

$$V = \text{upper bound of } S(I) \tag{1}$$

when I ranges over all possible subdivisions of (a, b).

It is generally *not* true that

$$V = \lim_{\delta \to 0} S\{I(\delta)\}, \tag{2}$$

where $I(\delta)$ stands for subdivisions, such that $\max (x_i - x_{i-1}) < \delta$, $1 \leq i \leq n$. But (2) holds if the subdivisions $I(\delta)$ satisfy also the additional condition that the abscissae, where $f(x)$ has the greatest external saltus, are among the points of subdivision of $I(\delta)$. More precisely, we want to prove the following:

Theorem. *Given a function $f(x)$ of bounded variation in the interval (a, b), of total variation V and with the external saltus, in non-increasing order of the salti, at the abscissae $c_1, c_2, \cdots, c_n, \cdots$; to every $\epsilon > 0$, arbitrarily small, there exist two functions, $N(\epsilon) < \infty$ and $\delta(\epsilon) > 0$, such that $S(I) > V - \epsilon$ for any $I = I(N, \delta)$, among whose points of subdivision are the first N abscissae $c_1, c_2, \cdots, c_N$ and whose subintervals do not exceed δ.*

Some results of this kind have been found in particular cases (see, for instance, [1; 335, §246]), and related problems have been treated before (see, for instance, [2; 269–272], [3], [4; 86–87, §§5–7]), but we think that the following proof is not without interest because (a) it is valid in the most general case (even when the points of discontinuity with external saltus are everywhere dense) and (b) it makes no artificial distinctions, according to the finite or infinite number of discontinuities, with or without external saltus, but uses only general properties of functions of bounded variation. (I am very much indebted to Professor A. S. Besicovich for valuable advice, which permitted me to improve and shorten considerably the proof.) As far as we could find out, no such proof has yet been published.

In order to prove the theorem, we need the following:

Lemma. *If the function $f(x)$ is of bounded variation in (a, b), then to any point ξ of that interval, which is not an outer saltus point, there corresponds a finite function*

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