SETS SUBTENDING A CONSTANT ANGLE ON A CIRCLE

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1. **Introduction.** Let C be a closed circular area in the plane, C', its boundary, and K, a closed convex set in C which subtends at every point of C' the same angle α, $0 < \alpha < \pi$. By this is meant that, at each point P of C' the angle between the two extreme supporting half lines to K at P is equal to α. If K is a circular area concentric with C and of radius $\sin \frac{1}{2}\alpha$ times the radius of C, it does subtend the angle α on C'. The question arises then as to whether or not the fact that K subtends a constant angle on C' implies that K is such a circle. (This problem was suggested by Professor F. A. Valentine at a seminar given by the author.)

It is shown in the following that the answer depends on the nature of the angle α. Let $\beta = \pi - \alpha$; we shall call K a β-set if K subtends $\pi - \beta$ on C'. If β is an irrational multiple of π, or if $\beta = (m/n)\pi$ where m/n in its lowest terms has even numerator, the only β-set is the concentric circle of radius $\cos \frac{1}{2}\beta$. If β is any other angle between zero and π, there exist non-circular β-sets, and these can be constructed with a considerable degree of arbitrariness.

In the case where non-circular β-sets are possible, a number of extremal properties are found, involving their perimeters, diameters, and widths.

For facts and formulas relating to convex bodies, used but not proved, see [1].

2. **A necessary and sufficient condition for a β-set.** Let C be of radius 1 and centered at the origin of the x-y plane. Let $p(\theta)$ be the supporting function of K, that is, the distance from the origin to the supporting line normal to that half line issuing from the origin and making an angle θ with the x axis. It is easily verified in our case that K must contain the origin as an interior point and that K can have no points on C', and so $0 < p(\theta) < 1$. Let P be on C' and S_1, S_2 be the two supporting lines to K through P, which intersect in the angle $\pi - \beta$. If half lines R_1 and R_2 are drawn from O, normal to S_1 and S_2, respectively, one will make an angle θ with the x axis and the other, an angle $\theta + \beta$. The distances from the origin to the supporting lines are $p(\theta)$ and $p(\theta + \beta)$, and one is led to the relation

$$\cos^{-1} p(\theta) + \cos^{-1} p(\theta + \beta) = \beta. \tag{1}$$

Here and henceforth the arc cosine will denote first quadrant angles. If in (1), θ is advanced by β, the result will, when subtracted from (1), yield $p(\theta) = p(\theta + 2\beta)$; that is, 2β is a period of $p(\theta)$. Now 2π is also a period of $p(\theta)$; hence, if β is an irrational multiple of π, $p(\theta)$ will have two incommensurable periods and, being continuous, will be constant. This makes K a circle with O as center.

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