INVARIANT THEORY OF THE GENERAL ORDINARY, LINEAR, HOMOGENEOUS, SECOND ORDER, DIFFERENTIAL BOUNDARY PROBLEM

By Thomas C. Doyle

This paper is concerned with an invariantive formulation of the boundary value problem treated in the first Herbert Ellsworth Slaught memorial paper [2].

1. The basic differential form. The differential equations,

$$y'' + p^2 y = 0, \qquad y'' + y'/x + (-p^2/x^2 + 1)y = 0,$$

$$y'' - (2x/(1 - x^2))y' + (p(p + 1)/(1 - x^2))y = 0,$$

admit as solutions trigonometric, Bessel and Legendre functions respectively. In terms of these solutions any function $f(x)$, arbitrary to within certain restrictions, may be expanded in a Fourier series, a Bessel series or a Legendre series respectively. A generalization of this idea leads to a consideration of the general differential equation with a complex parameter λ,

$$L(y, \lambda) = y'' + p(x)y' + (q(x)\lambda + r(x))y = 0 \qquad (q(x) \not\equiv 0). \tag{1.1}$$

Our starting point will be a study of the differential form $L(y, \lambda)$ under the most general [3] transformation group $G_2 : y = e^{\sigma(x)}y^*$, $x^* = \xi(x)$, $\xi'(x) \not\equiv 0$, preserving its form. Using the notation $y' = dy/dx$, $y^{*\prime} = dy^*/dx^*$, we obtain

$$y = e^{\sigma}y^*, \qquad y' = e^{\sigma}(\xi')^2((\xi')^{-1}y^{*\prime} + (\xi')^{-2}\sigma' y^*),$$

$$y'' = e^{\sigma}(\xi')^2\{y^{*\prime\prime} + (\xi')^{-1}(2\sigma' + (\xi')^{-1}\xi'')y^{*\prime} + (\xi')^{-2}(\sigma'' + (\sigma')^2)y^*\},$$

$$y'' + py' + (q\lambda + r)y = e^{\sigma}(\xi')^2(y^{*\prime\prime} + p^*y^{*\prime} + (q^*\lambda^* + r^*)y^*), \tag{1.2}$$

$$p^* = (\xi')^{-1}(p + 2\sigma' + (\xi')^{-1}\xi''), \qquad q^* = (\xi')^{-2}q,$$

$$r^* = (\xi')^{-2}(r + p\sigma' + (\sigma')^2 + \sigma''), \; \lambda^* = \lambda.$$

It is the tensor point of view to regard $L(y, \lambda)$ as expressed in an accidental one of the totality of allowable coordinate systems related by G_2 . In this sense, any other expression $L^*(y^*, \lambda^*)$ with arguments related to those of $L(y, \lambda)$ by (1.2) serves equally well to represent the basic differential form, and all such related forms are to be regarded as equivalent representations of one and the same abstract entity.

Received March 4, 1949; presented to the American Mathematical Society, February 26, 1949.