EXPANSIONS IN BANACH SPACES

By Bernard R. Gelbaum

1. **Introduction.** Various questions concerning the existence and character of series expansions in Banach spaces will be discussed in what follows. Throughout, we shall assume a certain fundamental familiarity with the contents of Banach's book [2]. Where advisable, definitions and theorems of a more obscure nature will be quoted together with their sources in the literature.

In Chapter I the properties of special bases in c_0 and l are investigated. Examples of the following are given:

(a) absolute and non-absolute bases for c_0 ;

(b) retro- and non-retro-bases for l (see Definition 1);

(c) a basis for c_0 whose associated biorthogonal functionals fail to span l.

Chapter II contains a treatment of the relationships subsisting among complemented manifolds, projections and bases (see Definition 2). We show that to each projection on a complemented manifold there corresponds, for each basis of the manifold, a unique set of biorthogonal linear functionals which serve to define the projection in a natural manner. This and the concept of a retro-basis, which stems from the investigations of Chapter I, lead to some theorems on reflexivity. The chapter is concluded with a discussion of absolute and Toeplitz bases of various types.

Chapter I

1. Let E be a Banach space, E^* its conjugate space, E^{**} its second conjugate space, *etc.* The elements of E, E^*, E^{**}, E^{***}, will be denoted by x, X, f, F respectively. Since various types of weak convergence will occur in the following we note:

(a) X_n is said to converge *-weakly (to X) if $X_n(x)$ converges (to $X(x)$) for all x in E;

(b) x_n is said to converge weakly (to x) if $X(x_n)$ converges (to $X(x)$) for all X in E^*.

$[x_\lambda]$ and $[X_\lambda]$ will denote the linear closures of the sets $\{x_\lambda\}$ and $\{X_\lambda\}$. If W is an arbitrary subset of E, we shall denote by W^+ the set: $\{X \mid X(x) = 0$, for all x in $W\}$. Similarly for a W in E^*, W_+ will be the set: $\{x \mid X(x) = 0$, for all X in $W\}$. We shall say X and x are orthogonal if $X(x) = 0$.

2. Definition 1. A sequence of elements $\{x_n\}$ in E is called a *basis* for E if, for every x in E, there is a unique sequence of real numbers $\{a_n\}$ such that the

Received June 21, 1948; in revised form, June 7, 1949.