GENERALIZED DEDEKIND SUMS AND TRANSFORMATION FORMULAE OF CERTAIN LAMBERT SERIES

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1. **Introduction.** The exact formula of Rademacher [5] which expresses the partition function $p(n)$ as a convergent series contains the Dedekind sums

$$s(h, k) = \sum_{\mu=1}^{k-1} \mu k^{-1}(h\mu k^{-1} - [h\mu k^{-1}] - \tfrac{1}{2}) \tag{1.1}$$

in the coefficients $A_k(n)$, where h is a positive integer and $[x]$ is the greatest integer in x. These sums were studied by Dedekind [1] in connection with the theory of the modular form $\eta(\tau)$ and by Rademacher and Whiteman [7] more recently from an arithmetical standpoint, the most interesting property of these sums being their reciprocity law:

$$12s(h, k) + 12s(k, h) = -3 + h/k + k/h + 1/hk \qquad ((h, k) = 1). \tag{1.2}$$

In this paper, the sums are generalized by considering

$$s_p(h, k) = \sum_{\mu=1}^{k-1} (\mu/k)\overline{B}_p(h\mu/k), \tag{1.3}$$

where $\overline{B}_p(x)$ is the p-th Bernoulli function (defined below in (2.11)), and $s_1(h, k) = s(h, k)$. The interest in these generalized sums lies in the fact that they satisfy a reciprocity law for odd p, of which (1.2) is merely a special case. Also, the sums $s_p(h, k)$ are related to the functions

$$G_p(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} n^{-p}x^{mn} \tag{1.4}$$

much in the same way that $s(h, k)$ is related to $\eta(\tau)$, $\log \eta(\tau)$ being the same as $(\pi i\tau/12) - G_1(e^{2\pi i\tau})$. Using a technique developed by Rademacher [6], transformation formulas relating $G_p(e^{2\pi i\tau})$ to $G_p(e^{2\pi i\tau'})$ are obtained for odd p, where $\tau' = (a\tau + b)/(c\tau + d)$ is a modular substitution. The sums $s_p(h, k)$ appear in these formulas. The sums $s_p(h, k)$ are expressible as infinite series related to certain Lambert series, and, for odd $p \geq 1$, $s_p(h, k)$ is also seen to be the Abel sum of a divergent series.

2. **Reciprocity law for the generalized Dedekind sums.** We denote by $\overline{B}_p(x)$ the p-th Bernoulli function given by the Fourier expansion

$$\overline{B}_p(x) = -p!(2\pi i)^{-p} \sum_{m=-\infty}^{+\infty}{}' m^{-p}e^{2\pi imx}, \tag{2.11}$$

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