THE RIEMANN ZETA-FUNCTION

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1. The aim of this paper is to prove a formula for the expansion of the square of the Riemann zeta-function in terms of two single zeta-functions and a series of Bessel functions. More precisely I establish the equivalence of the function

$$\zeta^2(u) - \zeta(2u) - 2\zeta(2u-1)\Gamma(1-u)\Gamma(2u-1)/\Gamma(u) \tag{1.1}$$

and the series

$$\Gamma(1-u)(2\pi)^u 2^{-\frac{1}{2}} \sum_{n=1}^{\infty} \sigma_{1-2u}(n)(-1)^{n+1} n^{u-\frac{1}{2}} Y_{u-\frac{1}{2}}(\pi n) \tag{1.2}$$

for all but isolated values of u. Here $Y_\nu(x)$ denotes the Bessel function of the second kind of order ν, and $\sigma_r(n)$ has the customary meaning $\sum_{d|n} d^r$ the summation here being taken over all divisors of n. Series of the type of (1.2), with, however, Bessel functions of the first kind instead of the second kind, are of course known as Schlömilch series.

A peculiarity of the formula is that the series (1.2) does not converge for any value of u, but has instead a property which I term "generalized Abel summability". I say that the series $\sum_1^\infty a_n$ has the generalized Abel sum t if we have $\lim_{\delta\to 0} \{\sum_{n=1}^\infty a_n e^{-n\delta} - \psi(\delta)\} = t$, where $\psi(\delta)$ is some finite combination of powers of δ and $\log \delta^{-1}$ of the form

$$\psi(\delta) = \sum_{r=0}^{p} (\log \delta^{-1})^r \sum_{s=1}^{q} \lambda_{rs} \delta^{\mu_{rs}}. \tag{1.3}$$

Here r runs through zero or positive integral values, the quantities λ_{rs} , μ_{rs} are independent of δ and the μ_{rs} have any values, real or complex, other than zero. (In the case of functions whose behavior is more complicated than that of the zeta-function it may be necessary to consider other forms for $\psi(\delta)$. An example is given in a recent paper of mine [1].)

This state of affairs I denote by $\sum_1^\infty a_n \approx s$. The following statements, which will be used in the sequel, may be verified without difficulty: (i) $\sum_1^\infty a_n = s$ implies $\sum_1^\infty a_n \approx s$; (ii) $\sum_1^\infty a_n \approx s$ and $\sum_1^\infty b_n \approx t$ together imply $\sum_1^\infty (a_n + b_n) \approx s + t$.

2. The following result on this type of summability is sufficiently general for my present purpose.

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