A NOTE ON THE SUMMABILITY OF FORMAL SOLUTIONS OF LINEAR INTEGRAL EQUATIONS

By Richard Bellman

In a recent issue of this journal it was demonstrated by Büchner [1] that the method of successive approximations may be applied to solve the linear integral equation

$$(1) \qquad u = f(s) + \lambda \int_0^1 K(s, t)u(t)\, dt$$

even in cases where the parameter λ is larger in absolute value than the radius of convergence of the Liouville-Neumann solution,

$$(2) \qquad U = f(s) + \lambda \int_0^1 K(s, t)f(t)\, dt + \cdots .$$

This extension is made possible by considering the sequence of functions $(U_n(s))$ generated by the following relation:

$$(3) \qquad U_{n+1} = \theta u_n + (1 - \theta)\lambda \int_0^1 K(s, t)U_n(t)\, dt + (1 - \theta)f(s),$$

where the choice of θ depends upon the value of λ. It was pointed out by Mr. Lorentz (see [1]) that this was equivalent to applying Euler-Knopp summability to the original series (2).

Büchner's proof is valid for any continuous, not necessarily symmetric, kernel $K(s, t)$. Previously, the symmetric case had been considered by Wiarda [5]. We shall also assume that $f(s)$ is continuous. The results may easily be extended to cover L^2-theory, at the expense of introducing "almost everywhere" language.

In connection with Büchner's results, it may be worthwhile to indicate the following easily proven statement which the author has been in possession of for several years.

Theorem. *Let S be a regular summability method which sums the series $1 + z + z^2 + \cdots + z^n + \cdots$ to $1/(1 - z)$ in a set of points R of the z-plane. Then S sums the formal series* (2) *to the Fredholm solution of* (1) *provided that $K(s, t)$ is a continuous symmetric kernel, that λ does not equal a characteristic value of $K(s, t)$, and that λ belongs to a set of points T, determined by R. In particular, if S sums the series of* (3) *to $1/(1 - z)$ for all $z \neq 1$, then S sums* (2) *to the Fredholm solution for all λ not equal to characteristic values.*

Received May 12, 1948. The results presented in this paper were obtained in the course of research conducted under the sponsorship of the Office of Naval Research.