THE MATRIX EQUATION $AX = XB$

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1. **Introduction.** The matrix equation $AX = XB$ has been discussed by many writers (see [5], [4]). In this paper the matrix A is taken in *rational canonical* form and the matrix X is obtained in terms of parameters. The relation $A_0X_0 = X_0B$ is true if, and only if, $PA_0X_0 = PX_0B$ for every non-singular matrix P. This is equivalent to $AX = XB$ where $A = PA_0P^{-1}$ and $X = PX_0$. Hence if all solutions of $AX = XB$ are known, so are all solutions of $A_0X = XB$ if a non-singular matrix P is given so that $PA_0P^{-1} = A$.

In the latter part of the paper the set S of all matrices commutative with A is considered. If k is the number of invariant factors of $\lambda I - A$ and X is a matrix of S, there is a matrix polynomial $M(\lambda)$ of degree k associated with X such that if $F(\mu, \lambda) = | \mu I_k - M(\lambda) |$, then $F(X, A) = 0$.

If A_0 is a square matrix of order n, there exists a non-singular matrix P such that $PA_0P^{-1} = A = \operatorname{diag}\{A_1, A_2, \cdots, A_k\}$ [1; 105] where

$$(1) \qquad A_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{in_i} \end{bmatrix} \qquad (i = 1, 2, \cdots, k)$$

and $n_i \geq n_{i+1}$. The characteristic function of A_i is

$$(2) \qquad \phi_i(\lambda) = \lambda^{n_i} - (a_{i1} + a_{i2}\lambda + \cdots + a_{in_i}\lambda^{n_i-1}),$$

and $\phi_i(\lambda)$ is divisible by $\phi_{i+1}(\lambda)$. In the discussion to follow it will be assumed that A is in this form.

2. **Non-derogatory matrices.** The matrix A_0 is non-derogatory if $k = 1$. That is, the characteristic function of A_0 is also its minimum function. Each matrix A_i given by (1) is non-derogatory.

Consider the equation

$$(3) \qquad A_iX = XB$$

where B is a square matrix of order m and X is to be determined. If X satisfies (3) then $g(A_i)X = Xg(B)$ for every scalar polynomial $g(\lambda)$. The r-th row of

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