COMMUTATIVE RINGS WITH RESTRICTED MINIMUM CONDITION

By I. S. Cohen

Introduction. By a ring we shall always mean a commutative ring with identity element. We shall be dealing with the ascending and descending chain conditions and shall frequently use these in the form of the maximum or minimum condition. The latter, for example, states that in every non-vacuous set of ideals there exists one not containing any other ideal of the set. If for every ideal $\mathfrak{a} \neq (0)$ in $\mathfrak{R}$, $\mathfrak{R}/\mathfrak{a}$ satisfies the minimum condition, then $\mathfrak{R}$ will be called a ring with *restricted minimum condition*, or briefly an RM-ring. Note that a ring with minimum condition is an RM-ring. A ring satisfying the maximum condition will be called *Noetherian*; $\mathfrak{R}$ is surely Noetherian if $\mathfrak{R}/\mathfrak{a}$ is Noetherian for every ideal $\mathfrak{a} \neq (0)$.

We are concerned with various more or less related results on rings $\mathfrak{R}$ satisfying the restricted minimum condition. Not all of the results are new; in some cases we have merely given simpler proofs of known results. In §1 we prove that the restricted minimum condition implies the maximum. In §2 we consider the effect on the restricted minimum condition of passing to a ring integrally dependent on $\mathfrak{R}$. In §3 we consider the factorization of the ideals of $\mathfrak{R}$ into prime ideals. In §4 we give a partial characterization of those rings in which every ideal has a basis of k elements, k independent of the ideal. The results of §4 are applied in §5 to so-called groups of finite rank.

The extent to which these results are already known is indicated in the individual sections.

1. **The maximum condition.** It is well known that in any ring (commutative or not) with identity, the minimum condition for left ideals implies the maximum condition for left ideals. This was proved by Hopkins [10] in 1939. It seems, however, not to have been generally noticed that this theorem was proved in the *commutative* case several years earlier by Akizuki [2]. We give in Theorem 1 a proof somewhat simpler than Akizuki's. It is to be noted that in contradistinction to the proof for the general case ([10; 726] or [11; 71]) the proof given here for the commutative case is relatively elementary; it uses, apart from elementary facts about rings, only the Jordan-Hölder theorem for commutative operator groups (including the fact that a composition series exists if and only if both chain conditions hold for admissible subgroups [11; 8]).

We recall our convention that "ring" means commutative ring with identity element. We note also that by a *proper* ideal in a ring $\mathfrak{R}$ is meant an ideal different from (0) and $\mathfrak{R}$.

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