A PROBLEM OF RAĬKOV

By Seymour Sherman

1. **Introduction.** Raĭkov, in an expanded version [5] of his thesis, considered the problem of developing harmonic analysis on commutative groups by appealing to the theory of group algebras rather than the structure of locally compact abelian groups, as is done in [7]. In setting up his group algebra, an appropriate invariant measure was a fundamental consideration and Raĭkov postulated an invariant measure (to which we refer as Haar-Raĭkov measure) on a topological group (not necessarily commutative). As a consequence of the postulates the group did not need to be locally compact but had to be locally totally bounded. In order to establish the usual properties of convolution, Raĭkov formulated a version of the Fubini Theorem, which involved an unusual σ-ring of sets. This led him to postulate his measure over the σ-ring $[B]_R$ of strict Borel sets.

The unusual mode of definition of $[B]_R$ suggested the question of whether the Haar-Raĭkov measure on locally totally bounded groups had the usual properties of Haar measure on locally compact groups. Raĭkov raised a number of these questions some of which we list here:

(1) Can Haar-Raĭkov measure be extended to the σ-ring of all Borel sets? As Raĭkov remarked, if the group were locally compact classical arguments show that the answer is yes. We show that in the locally totally bounded case the answer is also yes.

(2) What is the connection between Haar-Raĭkov measure on G and Haar measure on $\mathfrak{G}$, its completion? In particular can G be a Borel set of $\mathfrak{G}$ with Haar measure zero?

In the closely related case of a group H (no topology) with a "group measure", Weil [6] [7; Appendix I] has shown that this measure induces on H a topology (and uniform structure) so that its completion $\mathfrak{H}$ is locally compact and that the Haar measure on $\mathfrak{H}$ is closely related to the group measure on H. We show that the relation between the Haar measure of $\mathfrak{G}$ and Haar-Raĭkov measure on G is the same as that which holds between the Haar measure of $\mathfrak{H}$ and the "group measure" of H, [7; 145] and that in particular the outer Haar measure of G is not equal to zero.

The key to these questions is supplied by the recognition that the σ-ring [σ-bdd $B]_R$ of σ-bounded, strict Borel sets is the same as the σ-ring of Baire sets. This latter σ-ring (modulo countability assumptions on the group) arises in Kodaira's work [4; 77] and is also the part of $[B]_R$ which is significant for Raĭkov's applications to harmonic analysis. Another fundamental tool is a theorem of

Received November 10, 1948; in revised form, March 15, 1949. This paper was done under contract with the Office of Naval Research.