FIELDS OF PARALLEL VECTORS IN CONFORMALLY FLAT SPACES

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1. **Introduction.** In his paper on hypersurfaces in an Einstein space, Wong [12] has considered the problem of obtaining conditions on a conformally flat space C_n, $n > 3$, in order that it may admit a scalar ρ with vanishing second covariant derivative, $\rho_{,ij} = 0$. In case $\xi \equiv g^{ij}\rho_{,i}\rho_{,j} \neq 0$, he obtained the canonical fundamental form of such a C_n,

$$e_1(dx^1)^2 + \sum e_\alpha(dx^\alpha)^2/[1 + \tfrac{1}{4}K_0 \sum e_\alpha(x^\alpha)^2] \qquad (\alpha = 2, \cdots, n). \tag{1.1}$$

In case $\xi = 0$, the canonical form obtained was

$$\sum_1^{n-2} e_\alpha(dx^\alpha)^2 + 2dx^{n-1}\, dx^n + [Z \sum_1^{n-2} e_\alpha(x^\alpha)^2 + \sum_1^{n-2} Z_\alpha x^\alpha + Z_{n-1}](dx^n)^2. \tag{1.2}$$

In (1.1) the constant $K_0 \neq 0$, and in (1.2) the Z's are arbitrary functions of x^n. These results were based on the work of Brinkman [1].

The equations $\rho_{,ij} = 0$ imply the existence of a field of parallel vectors $\rho_{,i}$. In this paper we consider the problem of the existence of a set of $r = p + q$ fields of parallel vectors in a C_n, of which p are non-null, and q are null vectors. Such a C_n will be denoted by $C_n(p, q)$. It will be shown that if $r > 1$ the $C_n(p, q)$ is a flat-space, and hence the only possibilities are $C_n(1, 0)$ and $C_n(0, 1)$.

A $C_n(1, 0)$ is shown to be a symmetric space of Cartan of class 1. Also, both $C_n(1, 0)$ and $C_n(0, 1)$ are shown to be spaces of recurrent curvature, a type of space considered by Ruse in his study of harmonic spaces [9]. Such a space is a V_n in which (3.11) is satisfied, *i.e.*, a space of recurrent curvature.

Necessary and sufficient conditions in invariant form are stated in Theorem 3.1 in order that a C_n admit exactly one parallel vector field. Also new canonical forms of the fundamental forms are obtained in §4. In addition the equations of imbedding of a $C_n(1, 0)$ in a flat S_{n+1} are given, the $C_n(1, 0)$ being expressed as a spherical hypercylinder in the S_{n+1}. Finally some geometric properties of the $C_n(1, 0)$ are obtained involving its second fundamental form.

From a theorem of Cartan it is known that symmetric spaces admit transitive groups of motions with sub-groups of rotations [11; 48]. The group associated with a $C_n(1, 0)$ is found to consist of $1 + n(n - 1)/2$ parameters with a rotation sub-group of $n(n - 1)/2$ parameters.

Indices h, i, j, k, m have the range $1, \cdots, n$; α, β, the range $2, \cdots, n$; and other ranges are as indicated. We assume $n > 2$ unless otherwise stated.

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