AN OSCILLATION CRITERION INVOLVING A MINIMUM PRINCIPLE

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1. Let $q = q(x)$ be a real-valued continuous function on the half-line $0 \leq x < \infty$. A real-valued function $y = y(x)$, where $0 \leq x < \infty$, will be said to belong to class $\Omega(X)$, $X \geq 0$, if

(i) $y(x)$ is continuous for $0 \leq x < \infty$;

(ii) $y(x) = 0$ for $0 \leq x \leq X$;

(iii) the half-line $0 \leq x < \infty$ can be divided into a sequence of intervals $0 \leq x \leq a_1$, $a_1 \leq x \leq a_2$, $\cdots$, where $a_n \to \infty$ as $n \to \infty$, in such a way that $y(x)$ possesses a continuous derivative $y'(x)$ on each of these intervals;

(iv) $y(x)$ is of class (L^2) and is normalized by

$$\int_0^\infty y^2\, dx = 1; \tag{1}$$

and finally,

(v) the integral

$$\int_0^\infty (y'^2 + |q| y^2)\, dx < \infty. \tag{2}$$

It is clear that the class (of functions) $\Omega(x_1)$ contains the class $\Omega(x_2)$ if $x_1 \leq x_2$. Let $\mu = \mu(x)$ denote the greatest lower bound (g.l.b.), possibly $-\infty$, of the collection of numbers

$$J(y) = \int_0^\infty (y'^2 + qy^2)\, dx, \tag{3}$$

where y belongs to $\Omega(x)$, that is,

$$\mu(x) = \text{g.l.b. } J(y) \qquad (y \text{ in } \Omega(x)). \tag{4}$$

It is clear that $\mu(x)$ is a monotone non-decreasing function of x on $0 \leq x < \infty$ (with the understanding that possibly $\mu(x) \equiv -\infty$). The following oscillation criterion will be proved:

(*) *Let $q = q(x)$ be a continuous function on the half-line $0 \leq x < \infty$. The differential equation*

$$y'' - qy = 0 \tag{5}$$

is oscillatory, that is, every solution of (5) possesses an infinity of zeros on $0 \leq x < \infty$, if and only if the function $\mu(x)$ defined by (4) and (3) satisfies the inequality

$$\mu(x) < 0 \qquad (0 \leq x < \infty). \tag{6}$$

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