HARMONIC POLYNOMIALS

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1. **Introduction.** Most well-known characterizations of harmonic polynomials are linear in character; they depend upon variations of the Laplace equation or upon variations of the Gauss mean-value theorems. One recent exception is a remarkable result due to Gustin [3; 217] who used the non-linear character of his criterion to obtain simple proofs of some fundamental results in the theory of harmonic functions.

Characterizations of harmonic polynomials have been obtained in several recent papers; these criteria have been linear. It is the purpose of this note to obtain characterizations of harmonic polynomials which are analogues of Gustin's result; Gustin's theorem may be obtained from (6) below by introducing the integral implied by allowing n to tend toward infinity. We take this occasion to prove a characterization of harmonic polynomials due to Beckenbach and the present author [2].

2. **Definitions and lemma.** Let n denote a fixed integer, $n \geq 2$, and let ϕ denote an angle, $-\pi/n \leq \phi < \pi/n$. Then for each $r > 0$ the points $z + r\zeta_m \equiv (x + iy) + re^{i(\phi + \pi(2m-1)/n)}$, $m = 1, 2, \cdots, n$, are the vertices of a regular n-gon $p = p_n(z, r, \phi)$ with center z, circumradius r, and "orientation" ϕ. The length of p is $2nr \tan \pi n^{-1}$ and will be denoted by $\| p_n \|$.

For $n \geq 2$, and for each angle θ,

$$\sum_{m=1}^{n} [\cos(\theta + 2m\pi n^{-1}) + i \sin(\theta + 2m\pi n^{-1})]^2 = 0,$$

$$(1) \qquad \sum_{m=1}^{n} [\cos(\theta + 2mk\pi n^{-1}) + i \sin(\theta + 2mk\pi n^{-1})] = n\delta_{k,n}(\cos\theta + i \sin\theta),$$

where $\delta_{k,n} = 1$ if k is an integral multiple of n, and $\delta_{k,n} = 0$ otherwise (see [4; 924]).

The real and imaginary parts of $(x + iy)^n$ are basic homogeneous polynomials of degree n, for $n = 0, 1, 2, \cdots$. They will be denoted by $H_{1,n}(x, y)$ and $H_{2,n}(x, y)$, respectively.

If $f(x, y) \equiv f(z)$ is a function defined for $z = x + iy$ in the unit disc $\mathfrak{D}$: $|z| < 1$, then the following result holds (see [1; 336]).

Lemma. *If $f(x, y) \equiv f(z)$ is real and continuous in $\mathfrak{D}$, if n is a fixed integer, $n \geq 2$, and if ϕ is fixed, $-\pi/n \leq \phi < \pi/n$, then a necessary and sufficient condition that*

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