FOURIER SERIES OF L_2-FUNCTIONS

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1. **Introduction.** We will prove the following theorem.

If $f(x)$ is of period 2π and belongs to the Lebesgue class L_2, so that it has the Fourier expansion, $f(x) \sim \sum_{n=-\infty}^{\infty} a_n e^{inx}$, and if, for a fixed number x, we put

$$S^\delta(R) = \sum_{n^2 \leq R^2} (1 - n^2/R^2)^\delta a_n e^{inx} \qquad (\delta \geq 0),$$

$$f_0(t) \equiv f_0(x, t) = \tfrac{1}{2}[f(x + t) + f(x - t)],$$

$$f_p(t) \equiv f_p(x, t) = \frac{2\Gamma(p + \frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(p)t^{2p-1}} \int_0^t (t^2 - u^2)^{p-1} f_0(u)\, du \qquad (p > 0),$$

then the assumption

$$\int_\lambda^{2\lambda} \{S^\delta(R)\}^2\, dR = o(\lambda) \qquad (\lambda \to \infty), \tag{1.1}$$

implies $\int_0^t \{f_{\delta+1}(u)\}^2\, du = o(t)$, $t \to 0$, for $\delta > 0$; and also for $\delta = 0$ provided condition (1.1) is assumed to hold uniformly in an interval $0 \leq \delta < \delta_0$, which latter condition is certainly fulfilled if we have $S^0(R) = o(1)$, $R \to \infty$.

This is a special case of a theorem in several variables which is the one we will establish. If $f(x_1, \cdots, x_k)$ is of period 2π in each variable and belongs to L_2 in $0 \leq x_r < 2\pi$, $r = 1, \cdots, k$, so that it has the Fourier expansion

$$f(x_1, \cdots, x_k) \sim \sum a_{n_1 \cdots n_k} e^{i(n_1x_1 + \cdots + n_kx_k)}$$

and if we put $\nu^2 = n_1^2 + \cdots + n_k^2$,

$$S^\delta(R) = \sum_{\nu^2 \leq R^2} (1 - \nu^2/R^2)^\delta a_{n_1 \cdots n_k} e^{i(n_1x_1 + \cdots + n_kx_k)},$$

$$f_0(t) \equiv f_0(x, t) = \Gamma(k/2)2^{-1}\pi^{-k/2} \int_\sigma f(x_1 + t\xi_1, \cdots, x_k + t\xi_k)\, d\sigma_\xi,$$

where σ is the sphere $\xi_1^2 + \cdots + \xi_k^2 = 1$ and $d\sigma_\xi$ is its $(k - 1)$-dimensional volume-element, and if, for $p > 0$,

$$f_p(t) \equiv f_p(x, t) = (2/B(p, k/2)t^{2p+k-2}) \int_0^t (t^2 - s^2)^{p-1} s^{k-1} f_0(s)\, ds,$$

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