RATIONAL VECTOR SPACES I

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1. **Introduction.** Most of the modern investigations of vector spaces restrict the scalar quantities to the real or complex field. In this paper a detailed study is made of vector spaces over the field of rational numbers. These spaces are called rational vector spaces, and if an abstract rational-valued inner product is defined on such a space, the resulting system is called a rational inner product space.

Infinite dimensional rational inner product spaces, both denumerable and non-denumerable, are studied, and in this connection it is shown that every infinite dimensional and denumerable rational inner product space possesses a basis which is both normal and orthogonal. But the major emphasis is given to the rational inner product space of finite dimension. In the study of the structure of these spaces applications are made of the powerful Minkowski-Hasse theorems on the invariants of a quadratic form under rational transformations and the moment problem of Stieltjes.

2. **Rational vector spaces.** Let $V = \{0, \xi, \eta, \cdots\}$ be a commutative group under addition and let $R = \{0, 1, a, b, \cdots\}$ be the rational field. If for every $\xi \in V$ and $a \in R$, there is defined an element ξa of V (called the *scalar product* of ξ and a) such that $(\xi + \eta)a = \xi a + \eta a$, $\xi(a + b) = \xi a + \xi b$, $\xi(ab) = (\xi a)b$, $\xi 1 = \xi$, then the group V is called a *vector space over* R. The terms *rational vector space* and *rational space* will be used as equivalents for *vector space over* R.

Suppose now that V is a rational space. A subset $V_1 \subset V$ is called a *subspace* of V in case V_1 is a subgroup of V, and V_1 is itself a vector space over R. If $X = \{\xi\}$ is a subset of V, we use the notation $X \circ R$ for the set of all finite sums $\sum \xi_i a_i$, with $\xi_i \in X$ and $a_i \in R$. This is a subspace of V, and is in fact the intersection of all subspaces V_1 of V containing X.

A subset $X = \{\xi\}$ of a rational space V is *linearly independent* if every finite set of distinct vectors $\xi_1, \cdots, \xi_n$ of X is linearly independent in the usual sense. Otherwise the set $X = \{\xi\}$ is said to be *linearly dependent*. We say $X = \{\xi\}$ is a *Hamel basis*, or *basis*, for a rational space V in case X is linearly independent and $V = X \circ R$ (see [6]). Two rational spaces V_1 and V_2 are said to be *isomorphic* if there exists a biunique correspondence $\xi \leftrightarrow \xi'$ between the elements of V_1 and V_2 such that $(\xi + \eta)' = \xi' + \eta'$ and $(\xi a)' = \xi' a$ for all $\xi, \eta \in V_1$, $\xi', \eta' \in V_2$, and $a \in R$.

The rational space V is *infinite dimensional* if there exists in it an infinite set, the elements of which are linearly independent, and *finite dimensional* whenever it has a finite basis $\xi_1, \cdots, \xi_n$. If V has a finite basis $\xi_1, \cdots, \xi_n$,

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