AN ANALOG OF AN IDENTITY DUE TO WILTON

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1. **Introduction.** The following relation was demonstrated by Wilton [5]:

Theorem 1. *Let* $s = \sigma + it$, $s' = \sigma' + it'$, $\sigma > -1$, $\sigma' > -1$, $\sigma + \sigma' > 0$. *Then*

$$\zeta(s)\zeta(s') - \zeta(s + s' - 1)((s - 1)^{-1} + (s' - 1)^{-1})$$
$$= 2(2\pi)^{s-1} \sum_{n=1}^{\infty} \sigma_{1-s-s'}(n)n^{s-1}s \int_{2n\pi}^{\infty} u^{-s-1} \sin u \, du + (s \rightleftarrows s'),$$

where $(s \rightleftarrows s')$ *represents a term similar to the first with* s *and* s' *interchanged.*

In particular, if $s = \frac{1}{2} + it$, $s' = \frac{1}{2} - it$, there results

$$| \zeta(\tfrac{1}{2} + it) |^2 + 4\zeta(0)(1 + 4t^2)^{-1}$$
$$(1.1) \qquad = 2(2\pi)^{-\frac{1}{2}+it} \sum_{n=1}^{\infty} d(n)n^{-\frac{1}{2}+it}(\tfrac{1}{2} + it) \int_{2n\pi}^{\infty} u^{-3/2-it} \sin u \, du$$
$$+ (\tfrac{1}{2} + it \rightleftarrows \tfrac{1}{2} - it).$$

The preceding formula can be used to ascertain the asymptotic behavior of the mean value

$$(1.2) \qquad \lim_{T\to\infty} (2T)^{-1} \int_{-T}^{T} | \zeta(\tfrac{1}{2} + it) |^2 \, dt,$$

as $T \to \infty$. In place of (1.2) it is easier to follow Wilton and treat

$$(1.3) \qquad \lim_{\delta\to 0} \delta \int_0^{\infty} e^{-\delta t} | \zeta(\tfrac{1}{2} + it) |^2 . dt.$$

The relationship between (1.2) and (1.3) is well known as a consequence of the fundamental Tauberian theorems of Hardy and Littlewood.

The purpose of this paper is to generalize Theorem 1, and incidentally obtain a generalization of (1.1). We shall prove

Theorem 2. *For* $\sigma > \frac{1}{4}$, $\sigma' > \frac{1}{4}$, $\sigma + \sigma' > 1$, *the following identity is valid*:

$$\zeta^2(s)\zeta^2(s') - \frac{\zeta'(s + s' - 1)}{s' - 1} + \zeta^2(s + s' - 1)\left\{\frac{1}{(s' - 1)^2} - \frac{2C}{1 - s'}\right\}$$
$$(1.4) \qquad - \frac{\zeta'(s + s' - 1)}{s - 1} + \zeta^2(s + s' - 1)\left\{\frac{1}{(s - 1)^2} - \frac{2C}{1 - s}\right\}$$
$$= 2 \sum_{r=1}^{\infty} r^{s'-1} \int_{r^{1/2}}^{\infty} \frac{M_0(4\pi u)}{u^{2s'-1}} \, du \left(\sum_{kn=r} \frac{d(k)\, d(n)}{n^{s+s'-1}}\right) + (s \rightleftarrows s').$$

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