SETS OF CONVERGENCE OF TAYLOR SERIES I

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1. **Introduction.** Let $\sum_n a_n z^n$ be a Taylor series of radius of convergence one, with $\sum_n | a_n | = \infty$ and $\lim_n a_n = 0$. We consider the point set M on the unit circle C, on which the series converges. As Landau [2; 13–14] points out, the cardinal number of the set of such Taylor series is $\mathfrak{c}$, while the cardinal number of the set of subsets of C is $\mathfrak{f}$; hence, there exists a set M on C such that no Taylor series converges on M and diverges on $C - M$. It follows that if a set M on C is such that some Taylor series converges on M and diverges on $C - M$, the set must have certain special properties.

Lusin [3] (see also Landau [2; 69–71]) has constructed a Taylor series whose coefficients tend to zero and which diverges on the entire unit circle C. Sierpiński (see Landau [2; 71]) has modified Lusin's example to obtain divergence at all points of C except one. For every closed arc A on C, Neder [5] constructed a Taylor series which converges on $C - A$ and whose partial sums are unbounded at every point of A. Mazurkiewicz [4] used Neder's example to prove the following proposition: *If M is a closed set on C, there exists a Taylor series which converges on M and diverges on $C - M$, and a Taylor series which diverges on M and converges on $C - M$.*

The present paper is devoted to the extension of these results. Its method is inspired by Lusin's example.

2. **Two lemmas.** The present section contains all the arithmetic that is needed for proving the theorems of this paper.

LEMMA A. *Let $\beta_1, \beta_2, \cdots, \beta_{m-1}$ be real numbers subject to the restriction $2\pi - \delta \geq \beta_1 \geq \beta_2 \geq \cdots \geq \beta_{m-1} \geq \delta > 0$, and let $\zeta_1, \zeta_2, \cdots, \zeta_m$ be complex numbers of unit modulus, satisfying the condition $\zeta_{\nu+1}/\zeta_\nu = e^{i\beta_\nu}$ ($\nu = 1, 2, \cdots, m - 1$). Then $| \sum_1^m \zeta_\nu | < K/\delta$, where K is a universal constant.*

To prove this lemma, let

$$B_\nu = 1/(1 - e^{i\beta_\nu}) = \tfrac{1}{2}(1 + i \cot \tfrac{1}{2}\beta_\nu) \qquad (\nu = 1, 2, \cdots, m - 1).$$

Then $\zeta_\nu = (\zeta_\nu - \zeta_{\nu+1})/(1 - e^{i\beta_\nu}) = B_\nu(\zeta_\nu - \zeta_{\nu+1})$; therefore

$$\sum_{\nu=1}^{m} \zeta_\nu = \sum_{\nu=1}^{m-1} B_\nu(\zeta_\nu - \zeta_{\nu+1}) + \zeta_m$$

$$= B_1\zeta_1 + \sum_{\nu=1}^{m-2} (B_{\nu+1} - B_\nu)\zeta_{\nu+1} + (1 - B_{m-1})\zeta_m ,$$

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