THE COEFFICIENT REGIONS OF SCHLICHT FUNCTIONS

By A. C. Schaeffer, M. Schiffer and D. C. Spencer

1. We denote by $\mathfrak{F}$ the family of all functions $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ which are regular and schlicht in the circle $|z| < 1$. Let $\mathfrak{V}_n$ be the $2n$-dimensional region composed of all points $(a_2, a_3, \cdots, a_{n+1})$ belonging to the different elements of $\mathfrak{F}$, and let $\mathfrak{S}_n$ be the boundary of $\mathfrak{V}_n$. Since the family $\mathfrak{F}$ is compact, $\mathfrak{V}_n$ is a closed domain. (A closed domain is a closed set which is the closure of a domain.)

The coefficient problem for schlicht functions is that of determining $\mathfrak{V}_n$—that is to say, $\mathfrak{S}_n$—for all values $n = 1, 2, \cdots$. This problem was first seriously considered by Peschl [7] in 1937. At that time the most penetrating characterization of schlicht functions was provided by the method of Löwner [6]; Peschl applied this method to the coefficient problem and obtained qualitative results and also the 2-dimensional region of (a_2, a_3) when both are real. In the following years variational methods were introduced into the theory of schlicht functions [2], [3], [4], [8], [9], [11], [14], [15], [16], [18] (the systematic development of the variational method began with the paper [14]) and their development has provided new tools for the investigation of schlicht functions. In particular, the coefficient problem has been studied using variational methods [10], [12], [13], [19].

Since there is some overlapping of results presented here with those in [13] and [17], the repetition should be justified. The results of this paper are subsequent to [13] (despite different dates of submission) and each is subsequent to [17]. One of our main purposes is to bring the various ideas together under a general scheme and for the sake of completeness and clarity a certain amount of repetition seems unavoidable.

A combination of Löwner's method with variational methods was proposed in [17] and Löwner's differential equation was used in a somewhat different way in the coefficient problem (see [13]). It turns out that there is quite a remarkable duality relationship between the two ways in which the Löwner method has been combined with the variational approach. We show that the method adopted in [17] leads to the theory of the characteristic curves of the partial differential equation defining the boundary $\mathfrak{S}_n$ of $\mathfrak{V}_n$. It is interesting from the methodological point of view that the method of variations leads immediately to the partial differential equation for $\mathfrak{S}_n$ and that the classical theory of characteristic curves leads then necessarily to Löwner's differential equation for the schlicht functions associated with them. The method of the characteristics enables us to connect every point on $\mathfrak{S}_n$ with the distinguished "Koebe" point

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