GENERALIZED BERNSTEIN POLYNOMIALS

By I. I. Hirschman, Jr. and D. V. Widder

1. **Introduction.** A classical theorem of Weierstrass states that every function $f(x)$ continuous for $0 \leq x \leq 1$ can be uniformly approximated there by polynomials. S. Bernstein has given an explicit method of effecting the approximation in terms of the functions

$$\lambda_{n,k}(x) = \binom{n}{k} x^k (1 - x)^{n-k} \qquad (k = 0, 1, \cdots, n; n = 0, 1, \cdots). \tag{1.1}$$

The Bernstein polynomial corresponding to a function $f(x)$ is defined as

$$B_n[f(x)] = \sum_{k=0}^{n} f(k/n)\lambda_{n,k}(x). \tag{1.2}$$

For a proof that $B_n[f(x)]$ tends uniformly to $f(x)$ on $0 \leq x \leq 1$ as $n \to \infty$ see, for example, [4; 152].

C. Müntz has shown that the polynomials of Weierstrass's theorem can be replaced by linear combinations of the functions x^{a_k}, provided only that $a_0 = 0$, $\lim_{k\to\infty} a_k = \infty$, and $\sum_1^\infty a_k^{-1} = \infty$. It is the purpose of the present note to exhibit a set of functions analogous to (1.2) in terms of which this more general approximation can be performed. We shall call them generalized Bernstein polynomials. These functions have arisen earlier in our work [2], but we make an independent approach here so as to have a new and simple proof of the theorem of Müntz.

2. **A set of exponential polynomials.** It is convenient to make an exponential change of variables, so that we are considering the approximation of a function $f(x)$ continuous on $-\infty \leq x \leq 0$ ($f(-\infty)$ exists) by exponential polynomials. Consider a sequence of numbers $0 = a_0 < a_1 < a_2 < \cdots$. For each positive integer n we define $H_{n,n}(x)$ as $e^{a_n x}$ and $H_{n,k}(x)$, $k = 0, 1, \cdots, n - 1$, as the unique solution of the differential system ($D = d/dx$)

$$(1 - Da_{k+1}^{-1})(1 - Da_{k+2}^{-1}) \cdots (1 - Da_n^{-1})y(x) = e^{a_k x}, \tag{2.1}$$

$$y(0) = y'(0) = \cdots = y^{(n-k-1)}(0) = 0. \tag{2.2}$$

It is easy to write down an explicit expression for this solution:

$$H_{n,k}(x) = \frac{a_{k+1}a_{k+2} \cdots a_n}{D_{n,k}} \begin{vmatrix} e^{a_k x} & 1 & a_k & \cdots & a_k^{n-k-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ e^{a_n x} & 1 & a_n & \cdots & a_n^{n-k-1} \end{vmatrix}, \tag{2.3}$$

Received February 12, 1949. Research supported in part by the Office of Naval Research under contract NR 043-052.