NORMED ALGEBRAS

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1. **Introduction.** The object of central interest in this paper is the C^*-algebra, which may be defined concretely as a uniformly closed self-adjoint algebra of bounded operators on Hilbert space, or abstractly as a Banach *-algebra for which the critical axioms are $\| xx^* \| = \| x \|^2$, and xx^* has a quasi-inverse. It is not known whether this final axiom is redundant. In any event, we may study algebras for which it is not assumed; following Rickart [20] we call these B^*-algebras. In §§7–9 we prove some structure theorems for B^*-algebras satisfying special assumptions, *e. g.*, discreteness of the structure space, complete continuity, and finally (Theorem 9.2) being "central" and satisfying a polynomial identity. Some progress is also made toward determining whether B^*-algebras are C^*; the problem is reduced to the primitive case and then solved if there are minimal ideals. A key result in all these investigations is the fact (Theorem 7.2) that a homomorphic image of a B^*-algebra is again B^*. The preceding part of the paper is devoted to various preparatory results on involutions, complete continuity, *etc.* Since there are important cases where one encounters Banach algebras without a unit element, we have nowhere in the paper assumed a unit. Moreover many of the theorems are proved for real as well as complex scalars.

2. **Definitions.** By a *normed algebra* we shall mean a normed linear space which is also an algebra such that

$$(1) \qquad \| xy \| \leq \| x \| \ \| y \|.$$

A *Banach algebra* is a complete normed algebra. When it is necessary for clarity, we shall specify whether real or complex scalars are in question. We do not assume a unit element, nor, if there is one, that its norm is 1 (though by (1) its norm is necessarily at least 1).

We follow Jacobson [11] in the use of the following terminology: quasi-inverse, quasi-regular, radical, primitive; we also use the notation $x \circ y = x + y + xy$ and x' for the quasi-inverse of x. In a Banach algebra all elements within the unit sphere are quasi-regular; in the terminology of [14] a Banach algebra is a Q-ring. Moreover a normed algebra is a Q-ring if it is a Q_r-ring; more generally this is true for any topological ring whose completion is a Q-ring.

If I is a closed two-sided ideal in a normed algebra A, then A/I is a normed algebra in the usual norm $\| a + I \| = \inf \| a + x \|$, $x \in I$. If A is a Q-ring or

Received November 20, 1948. The author is a fellow of the John Simon Guggenheim Memorial Foundation.