DENSE CONVEX SETS

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Introduction. We will show that for a subspace S of a normed linear space E, the property of having a dense complementary subspace (unlike that of having a closed complementary subspace) depends merely on the deficiency of S in E. By use of this result it is shown that if E is an infinite-dimensional Banach space and $\aleph$ is a cardinal number no greater than that of E, then E can be expressed as the union of $\aleph$ pairwise disjoint dense convex sets. (For $\aleph = 2$ this was proved by J. W. Tukey [3; 101].)

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Preliminary definitions. We first review some definitions and observations of Löwig [1; 18–19] and Mackey [2; 158–159]. A maximal linearly independent subset of a (real) linear system L^* is called a *Hamel basis* for L^*. Each system has a Hamel basis and all bases for a fixed L^* have the same cardinal number, which we denote by $d(L^*)$ and call the *dimension* of L^*. If L and S are subspaces of L^*, $L \cap S = \{\phi\}$, and $L + S = L^*$, then each is said to be a *complementary subspace* (or simply *complement*) of the other. (ϕ is the neutral element of L^*.) Each subspace has a complement and all complements in L^* of a fixed L have the same dimension, which we denote by $\delta_{L^*}(L)$ and call the *deficiency* of L in L^*. Then $d(L) + \delta_{L^*}(L) = d(L^*)$, and to each pair of cardinal numbers $\aleph^1$ and $\aleph^2$ whose sum is $d(L^*)$ there corresponds a subspace L of L^* such that $d(L) = \aleph^1$ and $\delta_{L^*}(L) = \aleph^2$.

If E is a linear space (*i.e.*, a linear system having an associated topology) $\Delta(E)$ will denote the smallest cardinal number of a collection $\{U_a \mid a \in A\}$ of open subsets of E such that if $x_a \in U_a$ for each $a \in A$ then $\{x_a \mid a \in A\} \oplus$ is dense in E. ($X \oplus$ is the linear hull of X, $X \oplus Y \equiv [X \cup Y] \oplus$, *etc.*)

A linear space E will be called a *$(T\alpha)$-space* if the function $y + rx \mid x \in E$ is continuous whenever $y \in E$ and $r \in \mathfrak{R}$ (the real number field).

Dense complements.

(1) *Suppose that E is a linear space in which no proper subspace has an interior point and L is a subspace of E such that $\delta_E(L) \geq \Delta(E)$. Then L has a dense complementary subspace.*

Proof. By hypothesis there is a family $\{U_a \mid a \in A\}$ of open sets having the property mentioned in the above definition of $\Delta(E)$ and a complement S

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