ANALYTIC FUNCTIONS IN THREE-DIMENSIONAL RIEMANNIAN SPACES

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Introduction. Several attempts have been made to extend the theory of analytic functions of a complex variable to a hypercomplex variable of more than two real components. The most remarkable one was undertaken during the last fifteen years by R. Fueter and his pupils who aimed at the establishment of a function-theory of the partial differential equation $\Delta\phi = 0$, mostly in four variables. (This work was published mostly in the Commentarii Mathematici Helvetici beginning with vol. 7, 1935.) When trying to appraise what has been achieved one has to bear in mind that such a theory must inevitably be less rich in concepts and theorems than the classical theory of functions. What certainly cannot be expected in the general case is the possibility of constructing analytic functions out of others by the ordinary algebraic processes. But just this possibility is one of the basic features of the classical theory which even gives rise to the term "functions of a complex variable".

It has been shown recently that all integrals of arbitrary linear partial differential equations of second order and elliptic type can be built up out of analytic functions, provided that the coefficients of that differential equation depend analytically on the variables. (Except for this general remark no detailed reference need be made. Latest bibliography is given in [1]. See also [4] and [5].) This is done by an integral operator which leaves essential properties of the analytic functions, such as the behavior in the neighborhood of a singular point, untouched. To follow this line of thought is, therefore, another attempt to extend the theory of functions although it would be more correct to say that the theory of functions of a complex variable would be utilized for partial differential equations. This possibility is not restricted to the case of two variables. Whittaker was the first to construct all regular solutions of $\Delta\phi = 0$ in three variables by means of analytic functions, and Bergman [2] and [3] developed this idea.

We wish to do the same here for general linear partial differential equations of second order and elliptic type in three variables, the coefficients of which are analytic functions of the variables. It will become clear, too, that the restriction in the number of variables could be dropped altogether. Nevertheless, it may be advisable to treat the most simple case first. We shall consider the differential equation

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi + h^\mu\nabla_\mu\phi + k\phi = 0 \tag{1}$$

in the Riemannian space with the fundamental tensor $g^{\mu\nu}$, ∇_μ being the covariant differentiation operator, h^μ an arbitrary contravariant vector, and k an arbitrary

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