COMMUTER SYSTEMS IN A RING WITH RADICAL

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The theory of commuter systems (see [1], [2], [3], [7], [8], [10], [13], [14], [15], [17], [18], [19], [20], [21]) in a ring, particularly that of dual correspondence between subrings and their commuters of the "Vollstaendigkeit" (in the sense of Shoda [19]) of subrings, has been developed by many authors in different cases and in different generalities, not to speak of literature on endomorphism rings (of abelian groups), nor the case of rings of operators on Hilbert spaces as in Neumann [16]. Commuter has been called often commutator, and sometimes centralizor. Not only is the theory of Noether-Brauer-Artin-Whaples (see [1]; it was given independently also by Azumaya [3]; see [13] and [14] too) on commuter systems in division and simple rings typical and fundamental, but the theories hitherto developed deal exclusively, as it seems to the writer, with commuters of subrings (mostly semisimple but sometimes non-semisimple [2], [8], [14], [15]) in a simple or semisimple ring (or its infinite-dimensional analogue). The present note is to consider commuter systems in general non-semisimple rings with minimum condition. It seems to the writer that at least the fundamental feature of the theory prevails here too. An important role is played by the notion of regular, or quasi-regular, module, which was used also rather effectively in a Galois theory for general rings with minimum condition given recently by the writer [13]. (See also [4].) The following does not cover the generalized Galois theory of Jacobson [10] for division rings, nor the case without minimum condition, of [12] and [14]. However, as for the latter it is not difficult to generalize the following so as to include it. As for the former the writer wants to present in a subsequent paper a theory which combines the following with the one given in [13], a generalized Galois theory for rings with radical.

0. Let R be, throughout in this note, a ring with unit element 1 and satisfying the minimum condition (whence the maximum condition) for ideals.

An R-right-module $\mathfrak{m}$ is called *regular* (see also the notion of quasi-regularity in §4 below) when a direct sum of a certain number, say v, of its copies is R-isomorphic to the direct sum of a certain number, say u, of the copies of the R-right-module R. The number $h = u/v$ we call the *rank* of $\mathfrak{m}$; that it is uniquely determined by $\mathfrak{m}$ is clear from the theorem of Krull-Remak-Schmidt.

The R-endomorphism ring R^* of $\mathfrak{m}$ is nothing but the commuter ring $V(R)$ of R in the absolute endomorphism ring of $\mathfrak{m}$; $R^* = V(R)$. We have (see [3; §1])

(0.1) *The R-endomorphism ring $R^* = V(R)$ of $\mathfrak{m}$, a regular R-right-module, has unit element and satisfies the minimum condition, and the R^*-right-module $\mathfrak{m}$ is also regular and its rank is equal to h^{-1}. The R^*-endomorphism ring of $\mathfrak{m}$ not only contains but also coincides with R; $V(V(R)) = R$.*

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