FUNCTIONS WHOSE FOURIER-STIELTJES COEFFICIENTS APPROACH ZERO

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The main purpose of this note is to prove Theorem 2, below, which concerns generalized Fourier-Stieltjes coefficients of a function of bounded variation. If $k_n = 0$, Theorem 2 concerns the ordinary Fourier-Stieltjes coefficients. In this case Theorem 2 is deducible from results of Rajchman [2], [3], Verblunsky [4], and Young [5]. Theorem 2 is used to give a short proof of essentially known results, Theorems 3 and 4. The proof of Theorem 2 is based on a simple application of a complex variable technique developed previously by the writers [1].

THEOREM 1. *Let $f(z)$ be regular in the half plane $x \geq 0$ and satisfy there $|f(z)| \leq g(y)\, e^{\pi|y|}$ where $g(y)$ is a bounded function such that $g(y) \to 0$ as $y \to +\infty$. Let k_n, where $n = 1, 2, \cdots$, be a bounded sequence of real numbers, and suppose $f(n + ik_n) \to 0$ as $n \to \infty$. Then $f(x) \to 0$ as $x \to +\infty$.*

Proof. Suppose on the contrary that there is an unbounded sequence x_j, $j = 1, 2, \cdots$, where $|f(x_j)| > q > 0$. Let $n_j = [x_j]$ and $f_j(z) = f(z + n_j + 1/2)$. By the Ascoli-Vitali convergence theorem there is a subsequence of the $f_j(z)$ which converges uniformly in $|z| \leq r$ to an entire function $F(z)$ which clearly satisfies $|F(z)| \leq g(y)\, e^{\pi|y|}$. Since $|f_j(x)| > q$ for some point in the interval $-1/2 \leq x \leq 1/2$, it follows that $F(z)$ is not identically zero. Suppose $|k_j| < b$; then $f_j(x)$ becomes arbitrarily small at some point of each line segment $x = n - 1/2$, $|y| \leq b$, $n = 0, \pm 1, \pm 2, \cdots$. It follows that $F(z)$ has a zero in each of these segments.

Let $N(r)$ be the number of zeros of $F(z)$ in the circle $|z| = r$. In the rectangle $|y| \leq b$, $|x| \leq n - 1/2$, $n = 1, 2, \cdots$, there are at least $2n$ zeros of $F(z)$; therefore, if $r \geq b$, $N(r) \geq 2[(r^2 - b^2)^{1/2} + 1/2] \geq 2[r - b^2/r + 1/2]$. Assume first that $F(0) \neq 0$, and let $J(r) = \int_0^r N(t)\, dt/t \geq \int_b^r N(t)\, dt/t$. It is obvious that $\int_b^r ([t + 1/2] - [t + 1/2 - b^2/t])\, dt/t = O(1)$. To estimate the first integral it is noted that $\int_{n-1/2}^{n+1/2} [t + 1/2]\, dt/t = \int_{n-1/2}^{n+1/2} n\, dt/t = 1 + O(n^{-2})$. Thus $\int_b^r [t + 1/2]\, dt/t = r + O(1)$. Hence, for some constant c

$$J(r) \geq 2r - c. \tag{1}$$

Jensen's theorem states that $J(r) = (2\pi)^{-1} \int_0^{2\pi} \log|F(re^{i\theta})|\, d\theta - \log|F(0)|$. Substituting the bound, $\log|F| \leq \pi r|\sin\theta| + \log|g|$ yields

$$J(r) \leq 2r + (2\pi)^{-1} \int_0^{2\pi} \log|g|\, d\theta - \log|F(0)|. \tag{2}$$

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