A THEOREM OF STONE AND VON NEUMANN

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1. **Introduction.** Stone in [15] has stated and indicated a proof of the following

THEOREM. *Let $\mathcal{H}$ be a Hilbert space and let $P_1, P_2, \cdots, P_n, Q_1, Q_2, \cdots, Q_n$ $(n = 1, 2, \cdots)$ denote the members of a set of $2n$ self adjoint operators in $\mathcal{H}$. For each real number σ and each $j = 1, 2, \cdots, n$ let $U_\sigma^j = \exp(i\sigma P_j)$ and $V_\sigma^j = \exp(i\sigma Q_j)$ where $i^2 = -1$. Suppose that this system of operators is irreducible in the sense that there is no proper closed subspace of $\mathcal{H}$ which is carried into itself by all of the U_σ^j and all of the V_σ^j. Suppose finally that the following permutation relations are satisfied. For all j, k, σ and τ, $U_\sigma^j U_\tau^k = U_\tau^k U_\sigma^j$ and $V_\sigma^j V_\tau^k = V_\tau^k V_\sigma^j$. For all j, k, σ and τ with $j \neq k$, $U_\sigma^j V_\tau^k = V_\tau^k U_\sigma^j$. For all j, σ and τ, $U_\sigma^j V_\tau^j = \exp(-i\sigma\tau) \cdot V_\tau^j U_\sigma^j$. Then there exists a one-to-one linear norm preserving transformation S of $\mathcal{H}$ onto the Hilbert space of all complex valued functions of summable square on Euclidean n space of such a nature that $SP_jS^{-1}f(x_1, x_2, \cdots, x_n) = i\partial f(x_1, x_2, \cdots, x_n)/\partial x_j$ and $SQ_jS^{-1}f(x_1, x_2, \cdots, x_n) = x_j f(x_1, x_2, \cdots, x_n)$.*

In [8] von Neumann has given a proof of a sharpened form of this result. He omits the hypothesis of irreducibility and proves that $\mathcal{H}$ is the direct sum of at most countably many invariant subspaces for each of which the above conclusion is valid.

Rellich in [12] has recently given a proof of a variant of Stone's theorem in which the assumed permutation relations involve the P_j and the Q_k directly.

In the present paper we prove two related theorems. One of these (Theorem 1) is about general separable locally compact Abelian groups and reduces to a theorem equivalent to the Stone-von Neumann result when the group is taken to be the additive group of Euclidean n space. It is possible to state Theorem 1 in such a form that with a slight change in the hypotheses it has meaning for locally compact groups which are not necessarily Abelian. While Theorem 1 is not a special case of the resulting Theorem 2 it is an easy consequence of Theorem 2 and a well-known theorem on the unitary representations of locally compact Abelian groups. The argument used in the proof of Theorem 2 is closely related to and was in part suggested by the argument given by Gelfand and Neumark [4] in analyzing the irreducible unitary representations of the non-commutative two parameter Lie group. In fact Theorem 2 may be used to obtain the irreducible unitary representations of the members of a class of groups which includes that discussed by Gelfand and Neumark. This and related matters we expect to discuss in another paper.

The statements of Theorems 1 and 2 will be found in §2, and their proofs in §§6 and 7. The other sections contain preliminary material of various kinds.

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