COMPLETELY CONTINUOUS ELEMENTS OF A NORMED RING

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1. The contents of this paper arise almost entirely from a study of a particular class of elements of a normed ring, namely those which are completely continuous considered as operators on the underlying Banach space. Though the concept of a completely continuous element is thus primarily topological and though we base much of our work on theorems on linear operators due to F. Riesz and Banach, our methods are predominantly algebraic in that once this notion and those theorems are accepted and restated in the language of ideal theory, we proceed as though we were doing simple algebra. Complete continuity of an element not in the radical turns out very nicely to take the place of the classical descending chain condition in enabling us to construct an idempotent in an extremely direct fashion. It is interesting to note that this result can also be deduced using what we may term almost completely analytical methods, in contradistinction with ours, which we have called algebraic, from a theorem of Lorch which he proves with the Cauchy integral as his principal tool.

It is always a temptation, given a theorem on Banach spaces, to see whether something analogous can be proved replacing the norm by the weak topology. Only a small measure of success was attained in making this attempt in the present paper; it seems likely that one of the theorems referred to above, and which was essential to our proof, is definitely not true in the weak case though a less stringent form of it generalizes to the weak topology quite directly, and that consequently an entirely different approach will be needed to obtain the result if it holds at all.

Section 2 is preliminary in nature. We summarize our notation, list some known theorems and develop a few general ideas for later use. Section 3 deals with completely continuous elements, and §4 develops the corresponding theorems, as far as possible, for the weak case.

2. Our notation is standard and we summarize it here only for the sake of completeness. If A and B are sets, $A \cup B$ and $A \cap B$ are their union and intersection respectively. $A \subseteq B$ means that A is a subset of B, and $A \subset B$ that A is a proper subset of B. Convergence is denoted by $\rightarrow$. The norm of x is written $\| x \|$, the set of elements satisfying a condition is $\{x \mid \text{condition}\}$, and in this notation the unit sphere is $\{x \mid \| x \| \leq 1\}$ and will be denoted by S. Topological

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