SUMMATION FORMULAE INVOLVING ARITHMETIC FUNCTIONS

BY L. MIRSKY

1. **Introduction.** In a recent publication [5] I sharpened Carlitz's estimate [1] of the sum

$$\sum_{n_1+\cdots+n_s=n} n_1^{\alpha_1} \cdots n_s^{\alpha_s} F_1(n_1) \cdots F_s(n_s).$$

In the present paper I shall use somewhat similar ideas to give a corresponding treatment for the "conjugate" sum

$$(1) \qquad \sum_{0<n\leq x} F_1(n+k_1) \cdots F_s(n+k_s),$$

where $F_1, \cdots, F_s$ are arithmetic functions satisfying certain general conditions, and $k_1, \cdots, k_s$ are given integers. I had previously studied [3] a special case of the sum (1), and some lemmas from [3] and [5] will again be made use of in the present investigation.

In §3 some preliminary results will be established; these will lead, in §4, to the principal theorems of the paper dealing with the asymptotic behavior, for $x \to \infty$, of the sum (1) under various assumptions made about the functions F_i. The remainder of the paper (§§5–7) is concerned with applications. In §5 some properties of power-free integers are considered. In §6 summation formulae involving a wide class of multiplicative functions are established, and are used, in §7, to obtain a number of results concerning Euler's function.

2. **Notation.** The following notation will be used throughout the paper.

If $P_1(z)$, $P_2(z)$ are two propositions concerning the variable z, then $P_1(z)$ $(P_2(z))$ means that $P_1(z)$ holds for *every* z for which $P_2(z)$ holds.

The O-notation refers, unless otherwise stated, to the passage $x \to \infty$. The O-constants depend at most on σ, α, β, η, $k_1, \cdots, k_s$, $r_1, \cdots, r_s$, the arbitrarily small positive number ϵ, and the functions which occur in the investigation.

It will be frequently convenient to write $a \equiv b\ (\cdot m)$ in place of $a \equiv b \pmod m$, and also $\sum \{\mathfrak{C}\} f(m, n, \cdots)$ in place of $\sum f(m, n, \cdots)$ summed over $\mathfrak{C}$; here $\mathfrak{C}$ stands for the set of conditions defining the range of summation. All variables in summations take positive values unless otherwise stated.

The highest common factor and the lowest common multiple of $n_1, \cdots, n_s$ will be denoted by $(n_1, \cdots, n_s)$ and $[n_1, \cdots, n_s]$ respectively.

The letters $k_1, \cdots, k_s$ denote $s \geq 1$ fixed distinct non-negative integers; we write $k = \max_{1\leq i\leq s} k_i$.

The symbol $E(n_1, \cdots, n_s)$ is defined as 1 or 0 according as the system of s simultaneous congruences in n,

Received September 7, 1948.