THE MEAN CONVERGENCE OF ORTHOGONAL SERIES. III

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1. **Introduction.** In the present paper I shall extend to series of Jacobi polynomials my earlier results on ultraspherical series [3]. Let $w(x) = (1 - x)^\alpha (1 + x)^\beta$, $\alpha \geq -\frac{1}{2}$, $\beta \geq -\frac{1}{2}$, and let $\{p_n(x)\}$ denote the corresponding set of polynomials orthonormal on $(-1, 1)$. Write

$$M(\alpha, \beta) = 4 \max \left\{ \frac{\alpha + 1}{2\alpha + 3}, \quad \frac{\beta + 1}{2\beta + 3} \right\},$$

$$m(\alpha, \beta) = 4 \min \left\{ \frac{\alpha + 1}{2\alpha + 1}, \quad \frac{\beta + 1}{2\beta + 1} \right\}.$$

The extensions are these:

Theorem A. *If $f(x)$ is measurable and satisfies the condition*

$$\int_{-1}^{1} | f(x) |^p (1 - x)^\alpha (1 + x)^\beta \, dx < \infty$$

for a value of p in the interval $M(\alpha, \beta) < p < m(\alpha, \beta)$, then the expansion of $f(x)$ in its Jacobi series $f(x) \sim \sum a_n p_n(x)$ converges to $f(x)$ in the weighted p-th mean:

$$\lim_{N \to \infty} \int_{-1}^{1} | f(x) - \sum_{0}^{N} a_n p_n(x) |^p (1 - x)^\alpha (1 + x)^\beta \, dx = 0.$$

Theorem B. *The preceding conclusion fails if $p < M(\alpha, \beta)$ or $p > m(\alpha, \beta)$.*

2. **Reduction of Theorem** A. To prove Theorem A it suffices [3; Theorem 6.1] to show that the kernels

$$K_{\pm}(x, y) = \left| \left[\left(\frac{1 - y^2}{1 - x^2} \right)^{\pm 1/4} \left(\frac{w(y)}{w(x)} \right)^{1/2 - 1/p} - 1 \right] (x - y)^{-1} \right|$$

have the property that the functions $\int_{-1}^{1} K_{\pm}(x, y) f(y) \, dy$ belong to $L^p(-1, 1)$ when $f(y)$ does. This criterion is satisfied if for every pair of non-negative functions $f(y)$ and $g(x)$ in L^p and $L^{p'}$ respectively the integrals $\int_{-1}^{1} \int_{-1}^{1} K_{\pm}(x, y) f(y) g(x) \, dx \, dy$ converge. Since the argument in the two cases is the same we confine ourselves to the kernel K_+. The double integral can be written

$$\int_{-1}^{1} \int_{-1}^{1} K_+(x, y)^{1/p'} g(x) \left(\frac{1 - y^2}{1 - x^2} \right)^{-1/pp'} K_+(x, y)^{1/p} f(y) \left(\frac{1 - y^2}{1 - x^2} \right)^{1/pp'} dx \, dy,$$

Received September 9, 1948.