BASIC SETS OF POLYNOMIALS. II

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A basic set of polynomials is a set such that every polynomial can be uniquely expressed as a finite linear combination of elements of the set; it is called effective in the circle $| z | < a$ if the formal expansion (as defined below) of every function regular in $| z | < a$ converges to $f(z)$, uniformly in every circle $| z | \leq a' < a$. This terminology is due to J. M. Whittaker (see, for example, [5]). Whittaker conjectured that, if a set is effective in $| z | < a$, a given function cannot have two different expansions in $| z | < a$, and this was proved by S. H. Doss [2]. Here I shall prove a theorem from which it follows, in particular, that the expansion is unique if the coefficients can be computed for every function regular in $| z | < a$; this is a more general result since, for example, the coefficients of the expansion of every function regular in $| z | \leq 1$ can be computed for the Lidstone set [5; 10], while the expansion converges only for a special class of entire functions.

To state the general result of this note requires some notation. Let the polynomials of the set be $p_n(z) = \sum_{k=0}^{\infty} p_{nk} z^k$, where $p_{nk} = 0$ for $k > k(n)$. As Whittaker has shown, the set is basic if and only if the matrix $P = (p_{nk})$ has a (unique) row-finite reciprocal $\Pi = (\pi_{kn})$, and then the coefficient c_n of $p_n(z)$ in the formal expansion of $f(z)$ is

$$c_n = \sum_{k=0}^{\infty} \pi_{kn} f^{(k)}(0)/k!. \tag{1}$$

The matrix Π satisfies

$$\sum_{k=0}^{\infty} \pi_{nk} p_{km} = \delta_{nm} \tag{2}$$

(Kronecker δ) and also

$$\sum_{k=0}^{\infty} p_{nk} \pi_{km} = \delta_{nm} , \tag{3}$$

since it is both a right and a left reciprocal.

Let us consider the formal power series

$$q_k(z) = \sum_{n=0}^{\infty} \pi_{nk} z^{-n-1}. \tag{4}$$

Then our main theorem is as follows.

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