THE ZEROS OF THE JACOBI POLYNOMIALS AND THE CORRESPONDING CHRISTOFFEL NUMBERS

By Louis H. Kanter

1. This paper deals with a study of the dependence of the zeros $x_\nu = x(\nu, n, \tau)$ of the Jacobi polynomials, x_ν real and distinct, $-1 < x_\nu < 1$, and the corresponding Christoffel numbers $\lambda_\nu = \lambda(\nu, n, \tau)$, $\lambda_\nu > 0$, on the parameter τ, assuming that ν and n are fixed. Most of our results are concerned with the special class of Jacobi polynomials known as the ultraspherical polynomials, but we also obtain some results of a more limited nature in connection with the Jacobi polynomials in general.

To facilitate exposition we shall introduce some results needed for reference in addition to explaining the notation we shall use.

The symbol π_n will mean an arbitrary polynomial of degree n at most. Further, if (a, b) is a finite or infinite interval, and if $l(x)$ is a polynomial of exact degree n, which vanishes at the distinct points $x_1, x_2, \cdots, x_n$ contained in (a, b), then

$$l_\nu(x) = l(x)/l'(x_\nu)(x - x_\nu) \qquad (\nu = 1, 2, \cdots, n)$$

will denote the fundamental polynomials of the Lagrange interpolation corresponding to the x_ν, $\nu = 1, 2, \cdots, n$ (see [1; 322]).

The polynomials

$$h_\nu(x) = \left\{1 - \frac{l''(x_\nu)}{l'(x_\nu)}(x - x_\nu)\right\}\{l_\nu(x)\}^2 = v_\nu(x)\{l_\nu(x)\}^2 \tag{1.1}$$

are the fundamental polynomials of the first kind of the "Hermite interpolation" corresponding to the x_ν, $\nu = 1, 2, \cdots, n$ (see [1; 323–324, (14.1.8)]).

It will be convenient to have the equation

$$\int_a^b w_\tau(x, \tau)\{l(x)\}^2/(x - x_\nu)\, dx = \lambda_\nu(\tau)\{l'(x_\nu)\}^2 x_\nu'(\tau), \tag{1.2}$$

where $l(x)$ is the orthogonal polynomial of exact degree n corresponding to the weight function $w(x, \tau)$ on the interval (a, b) (see [1; 112, (6.12.4)]).

Again, if $l(x)$ is the Jacobi polynomial, then we have (see [1; 332])

$$\frac{l''(x_\nu)}{l'(x_\nu)} = \frac{\alpha - \beta + (\alpha + \beta + 2)x_\nu}{1 - x_\nu^2} \qquad (\alpha > -1; \beta > -1). \tag{1.3}$$

In the case of the ultraspherical polynomials

$$\lambda_\nu(\lambda) = 2^{2-2\lambda}\frac{\pi\{\Gamma(\lambda)\}^{-2}\Gamma(n + 2\lambda)}{(1 - x_\nu^2)\Gamma(n + 1)}\{P_n^{(\lambda)\prime}(x_\nu)\}^{-2}, \tag{1.4}$$

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