THE ZEROS OF CERTAIN REAL RATIONAL AND MEROMORPHIC FUNCTIONS

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1. **Introduction.** Let us assume that $f(z)$ and $f_1(z)$ are real polynomials whose ratio $F_1(z) = f_1(z)/f(z)$ has a partial fraction development of the form

$$F_1(z) = \sum_{j=1}^{n} \gamma_j/(z - c_j) \tag{1.1}$$

involving the n distinct complex numbers $c_j = a_j + ib_j$ and the complex numbers $\gamma_j = \alpha_j + i\beta_j = m_j e^{i\mu_j}$ with $\alpha_j \neq 0$ for all j. If not all the c_j are real, we may assume that $b_j > 0$ for $1 \leq j \leq p \leq n/2$; $b_j = -b_{j-p}$ and $\beta_j = -\beta_{j-p}$ for $p < j \leq 2p$, and $b_j = \beta_j = \mu_j = 0$ for $j > 2p$. For convenience, let us also assume that $|\mu_j| < \pi/2$ and that the m_j are positive or negative. Knowing the location of the zeros c_j of $f(z)$, the values of the μ_j and the signs of the m_j, we wish to determine the location of the zeros of $f_1(z)$.

Among the polynomials $f_1(z)$ is $f'(z)$, the derivative of $f(z)$. But the location of the real zeros of $f'(z)$ relative to those of $f(z)$ is described by the well-known theorem of Rolle. The location of the non-real zeros of $f'(z)$ relative to those of $f(z)$ is described by the following theorem of Jensen [2]. (The first published proof is by Walsh [5].)

Each non-real zero of the derivative of a real polynomial $f(z)$ lies in at least one of the circles (called the Jensen circles of $f(z)$) *which have as diameters the line segments joining the pairs of conjugate imaginary zeros of $f(z)$.*

In the present paper we propose to obtain for the zeros of $f_1(z)$ some generalizations not only of Rolle's Theorem and Jensen's Theorem but also of certain supplementary theorems due to Walsh. In these generalizations we shall replace the Jensen circles of $f(z)$ by *the circles $K(c_j, \mu_j)$, $j = 1, 2, \cdots, p$, of $F_1(z)$* defined as follows. The circle $K(c_j, \mu_j)$ shall be the circle which passes through the conjugate imaginary zeros c_j and $c_j^* = c_{j+p}$ and which has its center on the real axis at the point $z = k_j$ such that the angle c_j^*, c_j, k_j is μ_j. That is,

$$k_j = a_j + b_j \tan \mu_j \qquad (k = 1, 2, \cdots, p). \tag{1.2}$$

Thus the Jensen circles of $f(z)$ are the circles $K(c_j, 0)$ of $F_1(z) = f'(z)/f(z)$.

In §2 we shall consider the location of the real zeros of $f_1(z)$ and in §3 the non-real zeros of $f_1(z)$. In §4 we shall extend our results to systems of rational functions of the form (1.1); in §5 to meromorphic functions of similar form and finally in §6 to real functions of the form $G_1(z) = A - B^2z + F_1(z)$.

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