AN EXTENSION OF RAMANUJAN'S SUM

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1. **Introduction.** In [9] Ramanujan studied the sum

$$c_q(n) = \sum_{(h,q)=1} e^{2n\pi hi/q}, \tag{1.1}$$

where q, n are positive integers and the summation is over a reduced residue system (mod q). Ramanujan deduced several interesting properties of this sum and made various applications of it. It has also been used by other writers in treating problems of an arithmetic nature [4], [5], [6; 137–141].

Ramanujan's sum may be generalized by writing

$$c_q^s(n) = \sum_{(h,q^s)_s=1} e^{2n\pi hi/q^s}, \tag{1.2}$$

where in this case h ranges over the non-negative integers $< q^s$ such that h and q^s have no common s-th power divisors other than 1. It is clear that when $s = 1$, (1.2) reduces to the Ramanujan sum (1.1). In §2 we give several results for the extended sum (1.2) corresponding to known results for the simpler sum. Among these are the factorability property of $c_q^s(n)$ as a function of q:

$$c_{qq'}^s(n) = c_q^s(n)c_{q'}^s(n) \tag{1.3}$$

where $(q, q') = 1$.

In §3 we consider an analogue of (1.2) in the ring of polynomials $GF[p^n, x]$. This sum is an extension of the Carlitz η-sum [2; (4.3)]. An application of this sum is given in §4 in finding the number of solutions of

$$F = \alpha_1 X_1^s Y_1 + \cdots + \alpha_\nu X_\nu^s Y_\nu, \tag{1.4}$$

where $F \in GF[p^n, x]$, $\alpha_i \in GF(p^n)$, and appropriate restrictions are made as to the degrees of the X_i and Y_i. This problem is an extension of the one considered in [3]. Finally in §5, a corresponding problem for the rational domain is discussed.

2. **Formulas for the rational case.** For positive integers a, b, c, s we place $(a, b)_s = c^s$ if c^s is the largest s-th power divisor of both a and b. We also define $\phi_s(d)$ to be the number of non-negative integers $< d^s$ such that $(a, d^s)_s = 1$ and call such a set an s-reduced residue system (mod d). The function $\phi_s(d)$ is the Jordan ϕ-function [8; 95–97], and has the properties

$$\phi_s(dd') = \phi_s(d)\phi_s(d') \qquad ((d, d') = 1), \tag{2.1}$$

$$\phi_s(n) = n^s \prod_{p_i \mid n} (1 - p_i^{-s}), \tag{2.2}$$

where the product is over primes p_i such that $p_i \mid n$.

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