CONVERGENCE IN AREA

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Introduction.

The purpose of this paper is to establish an extension of an important result on *convergence in length.* In a closed interval I, let there be given continuous functions $f(x)$, $f_n(x)$, such that $f_n(x) \rightarrow f(x)$ uniformly in I. Let $L(f, I)$, $L(f_n, I)$ denote the length of the curves $y = f(x)$, $y = f_n(x)$, $x \varepsilon I$, respectively. Let $V(f, I)$, $V(f_n, I)$ be the total variations of the functions $f(x)$, $f_n(x)$ respectively in the interval I. Suppose that $L(f, I) < \infty$, $L(f_n, I) < \infty$, $n = 1, 2, \cdots$. Adams and Lewy [1] have established the theorem: *if, in addition to the conditions already stated, we have the relation* $L(f_n, I) \rightarrow L(f, I)$, *then it follows that* $V(f_n, I) \rightarrow V(f, I)$. This interesting result can be extended readily to curves given in general parametric form. Let x_1, x_2, x_3 be Cartesian coordinates in Euclidean three-space. We shall use $\mathfrak{x}$ to denote the vector with components $[x_1, x_2, x_3]$. Similarly, if $x_1(u)$, $x_2(u)$, $x_3(u)$ are continuous functions on an interval I: $a \leq u \leq b$, then $\mathfrak{x}(u)$ denotes the vector function with components $[x_1(u), x_2(u), x_3(u)]$. Now let there be given, on a fixed interval I: $a \leq u \leq b$, continuous vector functions $\mathfrak{x}(u)$, $\mathfrak{x}_n(u)$, $n = 1, 2, \cdots$. Then the equations $\mathfrak{x} = \mathfrak{x}(u)$, $\mathfrak{x} = \mathfrak{x}_n(u)$, $u \varepsilon I$, may be considered as representations of curves C, C_n. Let us consider the curve

$$C: \qquad \mathfrak{x} = \mathfrak{x}(u) \qquad\qquad (u \varepsilon I).$$

We associate with C three curves C^1, C^2, C^3 defined as follows:

$$C^i: \qquad \mathfrak{x} = \mathfrak{x}_i(u) \qquad\qquad (u \varepsilon I),$$

where $\mathfrak{x}_1(u)$, $\mathfrak{x}_2(u)$, $\mathfrak{x}_3(u)$ are the vector functions with components $[x_1(u), 0, 0]$, $[0, x_2(u), 0]$, $[0, 0, x_3(u)]$ and, of course, $[x_1(u), x_2(u), x_3(u)]$ are the components of $\mathfrak{x}(u)$. The curves C^i, $j = 1, 2, 3$, may be considered as the *projections* of C upon the coordinate axes. Let $L(\mathfrak{x}, I)$ be the length of C, and similarly let $L(\mathfrak{x}_i, I)$ be the length of C^i. Clearly, $L(\mathfrak{x}_i, I)$ is merely the total variation in I of the component $x_i(u)$ of $\mathfrak{x}(u)$. Let the symbols C_n^i, $L(\mathfrak{x}_n, I)$, $L(\mathfrak{x}_{ni}, I)$ have analogous meaning relative to the curves $C_n: \mathfrak{x} = \mathfrak{x}_n(u)$, $u \varepsilon I$. Suppose that $L(\mathfrak{x}, I) < \infty$, $L(\mathfrak{x}_n, I) < \infty$. We have then the following plausible generalization of the theorem of Adams and Lewy: if $\mathfrak{x}_n(u) \rightarrow \mathfrak{x}(u)$ uniformly on I and $L(\mathfrak{x}_n, I) \rightarrow L(\mathfrak{x}, I)$, then $L(\mathfrak{x}_{ni}, I) \rightarrow L(\mathfrak{x}_i, I)$, $j = 1, 2, 3$ (see [3]). It is natural to ask if analogous theorems hold for surfaces. The case of surfaces given in

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