# FIELDS OF PARALLEL VECTORS IN PROJECTIVELY FLAT SPACES

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1. **Introduction.** The problem of obtaining canonical forms for the components $\Gamma^i_{jk}$ of the symmetric affine connection of a projectively flat space $P_n$ admitting $p$ fields of contravariant or covariant vectors is solved, and the forms of the corresponding vector components are also obtained. In §§2 and 3 in which the contravariant vector problem is considered it is shown that the contracted curvature tensor with components $B_{ij}$ must be symmetric. The canonical forms are expressed in a coordinate system $(x^i)$ for which (see [1; 96])

$$\Gamma^i_{jk} = -(\delta^i_j \psi_k + \delta^i_k \psi_j), \tag{1.1}$$

where $\psi_i$ are components of an arbitrary vector.

If $B_{ij}$ is symmetric, $\psi_i$ is a gradient $\partial\psi/\partial x^i$. There are two possible canonical forms for $e^\psi$, one being an arbitrary function homogeneous of degree one in $x^p, x^{p+1}, \cdots, x^n$, the other being an arbitrary function of $x^{p+1}, \cdots, x^n$. The associated components of the parallel contravariant vectors are given in Theorem 3.1.

In §§4 and 5 the covariant vector problem is treated. It is shown that the number of parallel fields must be either $p = 1$, or $p = n$, the latter case corresponding to a flat space. For $p = 1$, the covariant vector $\lambda_i$ is the gradient of a function $\lambda$, and the equation defining $\lambda$ in the coordinate system of (1.1) represents a ruled hypersurface in the $(n + 1)$-dimensional Euclidean space with coordinates $(x^1, \cdots, x^n, \lambda)$, the rulings being $(n - 1)$-dimensional linear varieties parallel to the hyperplane $\lambda = 0$.

Section 5 treats the symmetric case $B_{ij} = B_{ji}$. In this case the ruled hypersurfaces are either generalized right conoids or hypercylinders.

In §6 some necessary conditions in invariant form are obtained for the existence of parallel covariant fields. Necessary and sufficient conditions in invariant form for either type of parallel field remain as an unsolved problem.

Small italic indices will have the range $1, 2, 3, \cdots, n$; the indices $\alpha, \beta$ the range $1, \cdots, p$. Special ranges are indicated where used.

2. **Contravariant fields of parallel vectors, $n > 2$.** A $P_n$ is characterized by the vanishing of the Weyl tensor with components (see [1; 89])

$$\begin{aligned} W^h_{ijk} &= B^h_{ijk} + \frac{1}{n+1}\,\delta^h_i(B_{jk} - B_{kj}) \\ &\quad + \frac{1}{n^2-1}\Big[\delta^h_j(nB_{ik} + B_{ki}) - \delta^h_k(nB_{ij} + B_{ji})\Big] = 0, \end{aligned} \tag{2.1}$$

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