GROUPS WITH DESCENDING CHAIN CONDITION FOR NORMAL SUBGROUPS

By Reinhold Baer

The existence of infinite simple groups shows that the class of groups under discussion is far removed from the class of finite groups; and the example of the (abelian) groups of type p^∞ shows that we cannot expect the double chain condition for normal subgroups to be a consequence of the descending chain condition for normal subgroups.

Still one may ask whether there are not at least some limitations on the length of ascending chains and whether we cannot prove finiteness of suitable parts of such groups. The most obvious example of such a theorem is the fact that the intersection of all the normal subgroups of finite index will itself have finite index; and deeper, but typical, is the fact that the ascending central chain will always terminate after less than $\omega 2$ steps.

The principal objective of this investigation is the search for similar phenomena. Most of our discussion is concentrated around the following concepts: Let G be a group such that the descending chain condition is satisfied by its normal subgroups. Define inductively: $S_F(G, 0) = 0$, $S_F(G, \nu + 1)$ is the uniquely determined normal subgroup of G which contains $S_F(G, \nu)$ such that $S_F(G, \nu + 1)/S_F(G, \nu)$ is the sum of all the finite minimal normal subgroups of $G/S_F(G, \nu)$, and $S_F(G, \rho)$, for ρ a limit ordinal, is the join of all the $S_F(G, \nu)$ with $\nu < \rho$. Then we may show that this series terminates after less than $\omega 2$ steps, that $S_F(G, \omega 2)/S_F(G, \omega)$ is finite and that $S_F(G, \omega)$ is exactly the set of all the elements in G which possess only a finite number of conjugates in G. Furthermore $S_F(G, \omega)$ contains a normal subgroup A of G such that A has finite index in $S_F(G, \omega)$, and is the direct sum of a finite number of groups of type p^∞. Thus the two examples with which we began our discussion really constitute the major step from finite groups to the groups with descending chain condition for normal subgroups.

The author is gratefully indebted to Miss Christine Williams for patiently reading various drafts of this paper, for her critical comments and valuable suggestions.

Notations. The composition of the group elements will be designated as addition $x + y$. If A and B are subsets of the group G, then $A \cap B$ = cross cut of A and B; $[A, B]$ = subgroup generated by all the commutators $-a - b + a + b$ for a in A and b in B; AB = set of all elements $ab = -b + a + b$ for a in A and b in B; $\{A\}$ = subgroup generated by A; $C(A) = C(A < G)$ = centralizer of A in G = set of all the elements c in G such that $c + a = a + c$ for every a in A. $Z(G) = C(G < G)$ = center of the group G. $G \simeq H$ signifies that G and H are isomorphic groups.

Received May 12, 1948.