OSCILLATION THEOREMS FOR SELF-ADJOINT BOUNDARY VALUE PROBLEMS

BY MORRIS J. GOTTLIEB

1. **Introduction.** The self-adjoint boundary value problem corresponding to the Sturm-Liouville equation

$$d(R(x, \lambda)y')/dx + P(x, \lambda)y = 0 \qquad (a \le x \le b), \tag{1.1}$$

with the self-adjoint boundary conditions

$$\begin{aligned} a_1y(a) + b_1z(a) &= c_1y(b) + d_1z(b), \\ a_2y(a) + b_2z(a) &= c_2y(b) + d_2z(b) \end{aligned} \tag{1.2}$$

has been discussed in the literature since 1836 (see [10]). Here, $y' = dy/dx$, $z = R(x, \lambda)y'$. The condition of self-adjointness is that the rank of the matrix

$$(a_i, b_i, c_i, d_i) \equiv \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{pmatrix}$$

is two, and that the determinant equation

$$| a_i, b_i | = | c_i, d_i | \tag{1.3}$$

holds. The coefficients of the matrix (a_i, b_i, c_i, d_i) are assumed to be functions of λ subject to suitable restrictions (see §2.2).

Bôcher [1] discusses the classical Sturm case where the conditions reduce to conditions on each end-point separately, as well as the periodic case. Haupt [4] and Ettlinger [2], [3] give results on the oscillations of solutions of the boundary value problem in the general case. Kamke [5] extends the ingenious method of Prüfer [8] to obtain oscillation theorems in the general case. An excellent account of the applications of the methods of the calculus of variations to this problem by Hilbert, Courant, Bliss, Morse, Birkhoff and Hestenes, and others is given in an article by Reid [9] which also contains a comprehensive bibliography.

In this paper we apply Morse's representation of self-adjoint boundary conditions [6; 83] and methods based on the index form [6] to make a complete and systematic classification of boundary value problems of the type described in §1 and to derive suitable oscillation theorems for each class. We assume that R, P, and $\partial R/\partial x$ are continuous in x and in λ and that $R(x, \lambda) > 0$ for λ real and x in [a, b]. We use the notation (a, b) for the interval $a < x < b$, $[a, b]$ for $a \le x \le b$, $(a, b]$ for $a < x \le b$, *etc.*

Received June 10, 1948. This work was done on a U. S. Navy contract under the supervision of Professor Marston Morse at the Institute for Advanced Study.