BOUNDS FOR CHARACTERISTIC ROOTS OF MATRICES

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In a recent note A. Brauer [1] considered the bounds for the characteristic roots of matrices with complex coefficients. He was able to show that all the roots lie inside or on the boundary of the region formed by certain circles associated with the matrix in a simple manner. It will now be shown that the position of the roots inside this region can be described in more detail. The application of these ideas to numerical cases is described elsewhere [2].

Brauer's method is based on the fact that an $n \times n$ determinant $| u_{ik} |$ with

$$| u_{ii} | > \sum_{\substack{k=1 \\ k \neq i}}^{n} | u_{ik} | \qquad (i = 1, \cdots, n) \tag{1}$$

is different from zero.

Applying this to a characteristic determinant $| a_{ik} - \lambda\delta_{ik} |$ it follows that the roots λ lie inside or on the boundary of the n circles C_i with centers a_{ii} and radii

$$\sum_{\substack{k=1 \\ k \neq i}}^{n} | a_{ik} |.$$

This statement can immediately be made more precise because the determinant theorem still holds if some, but not all, of the inequalities (1) are replaced by equalities—the only exception being the matrices which can be reduced to the form $\begin{pmatrix} P & 0 \\ U & Q \end{pmatrix}$ by the same permutation of the rows and columns. (See [3].) Here P and Q are square matrices and 0 consists of zeros.

Such a reduction will be possible or impossible simultaneously for a matrix and its characteristic matrix. The reduced matrix will have the same circles corresponding to it, only in a different order. Also, the roots of the matrix are left invariant since the reduction can be carried out by transforming the original matrix by a non-singular matrix. Applying the generalized determinant theorem to the characteristic determinant the following result is obtained.

Assume that the matrix cannot be reduced to the form $\begin{pmatrix} P & 0 \\ U & Q \end{pmatrix}$ *by the same permutation of the rows and columns. A characteristic root can then be a boundary point of the region formed by the circles* C_i *only if it is a common boundary point of all the circles.*

The case $n = 2$ will now be studied in greater detail. The following two theorems hold if the circles are non-degenerate.

Theorem 1. *A point common to both circles cannot be a root unless it is a common boundary point.*

Theorem 2. *A double root cannot be a common boundary point unless the circles touch and have equal radii.*

Received January 23, 1948; in revised form, July 14, 1948.