ANALYTIC FUNCTIONS POSSESSING A POSITIVE REAL PART

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In 1947, Ahlfors [1] proved the following theorem: Let D be a domain of connectivity n in the complex z-plane containing $z = \infty$, and consider the family B of analytic functions $f(z)$ which are regular and single-valued in D and satisfy there $|f(z)| \leq 1$ and $f(\infty) = 0$. If the expansion of $f(z)$ in the neighborhood of $z = \infty$ is $f(z) = b_1 z^{-1} + b_2 z^{-2} + \cdots$, then $|b_1|$ is maximized, within the family B, by a function $w = f_0(z)$ which maps D onto a domain which covers the entire circle $|w| < 1$ exactly n times.

Using the transformation $w' = (1 + w)(1 - w)^{-1}$, this theorem may also be formulated as a result for the coefficients a_1 of the family P of functions $f(z) = 1 + a_1 z^{-1} + a_2 z^{-2} + \cdots$ which are regular and single-valued in D and have there a positive real part. Ahlfors' result suggests that we look for a similar geometrical characterization of the functions, which, within the family P, maximize a_m, $m = 2, 3, \cdots$. Since the domain of variability of the coefficient a_m will, for $m > 1$, in general not be a circle, we shall have to consider the more general problem $\operatorname{Re}\{e^{i\theta} a_m\} = \max$, $0 \leq \theta < 2\pi$, if we wish to obtain all the functions which correspond to boundary points of the domain of variability of a_m.

It will turn out that the mappings effected by the extremal functions of this and some related problems are all of the same type as in the case considered by Ahlfors. In order to avoid repeated reference to the properties of the extremal maps, we shall use the following definition: *A domain of type T* is an n-fold connected domain in the w-plane which covers every point w for which $\operatorname{Re}\{w\} > 0$ exactly n times, and does not contain any points for which $\operatorname{Re}\{w\} < 0$. Obviously, its n boundary curves all coincide with the imaginary axis. A domain of type T may also be described as one "half" of an n-sheeted closed Riemann surface of genus $n - 1$, which is symmetrical with regard to the imaginary axis. It is well known [2], [3], [4] that any n-fold connected domain D may be mapped onto a domain of type T; it is, moreover, possible to assign n arbitrary points P_ν, one on each boundary continuum of D, and to construct a mapping onto a domain of type T such that the conformal images of the points P_ν all coincide with the same point on the imaginary axis. The mapping function will then be uniquely determined up to a linear transformation of the half-plane $\operatorname{Re}\{w\} > 0$ into itself.

After these preparations, we state the following theorem.

Theorem 1. *Let D be an n-fold connected domain in the z-plane containing the point $z = \infty$, and let P denote the family of functions $w = f(z)$ which are regular and single-valued in D, have there a positive real part, that is,* $\operatorname{Re}\{f(z)\} \geq 0$ *for*

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