FINITENESS PROPERTIES OF GROUPS

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Every finite group G has the following three properties.

(FC) Every element in the group G possesses only a finite number of conjugates in G.

(LF) Every element in the group G is contained in a finite normal subgroup of G.

(FO) There exists only a finite number of elements of any given order in the group G.

On the other hand it is clear that there exist infinite groups which have one or all of these properties. Thus we propose here to characterize the groups with these properties and to investigate the interrelations among these properties.

It is clear that LF-groups as well as FO-groups are FC-groups without elements of order 0 (= infinite order). It is not so obvious and will be shown here that a group G is an LF-group if, and only if, G is an FC-group without elements of order 0. From this theorem one deduces furthermore that FO-groups are LF-groups. Thus Property (FO) implies Property (LF) and Property (LF) implies Property (FC); and it is quite obvious that neither of the converses is true. But the group G is an FC-group if, and only if, each of its elements is contained in a finitely generated normal subgroup of G and its central quotient group is an LF-group, a theorem which shows that FC-groups and LF-groups are not too far apart. That the FO-groups are quite close to finite groups may be seen from the following characterization: The group G is an FO-group if, and only if, its center is an FO-group and its center quotient group contains, for every prime p, only a finite number of elements of order a power of p.

Some of the results obtained here will be needed in another context in somewhat greater generality than would be necessary for our present purposes; and thus we derive them immediately in the desired generality.

Notations. Composition of group elements will be written as addition $x + y$. If T is a subset of the group G, then $\{T\}$ = subgroup generated by T; $C(T < G)$ = centralizer of T in G = set of all elements z in G such that $t + z = z + t$ for every t in T.

If S and T are subsets of the group G, then $S \cap T$ = cross cut of S and T; ST = set of all elements $-t + s + t = st$ for s in S and t in T.

$Z(G) = C(G < G)$ = center of G.

The element z in G has order 0 if $nz = 0$ implies $n = 0$. If z is not of order 0, then z has positive order.

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