SOME PROPERTIES OF SUMMATION KERNELS

By Richard Bellman

1. **Introduction.** In this paper we wish to consider various properties of summation kernels having the general form

$$K(t, \theta) = \sum_{n=0}^{\infty} e^{-\lambda_n t} \begin{Bmatrix} \cos n\theta \\ \sin n\theta \end{Bmatrix} \qquad (\lambda_{n+1} > \lambda_n > 0). \tag{1}$$

The following cases of the above will be discussed:

$$\begin{aligned}
&\text{(a) } (\lambda_n = n) \quad & P(t, \theta) &= 1 + 2 \sum_{n=1}^{\infty} e^{-nt} \cos n\theta \\
& & &= (1 - e^{-2t})/(1 - e^{-2t} \cos \theta + e^{-2t}) \qquad (t > 0); \\
&\text{(b) } (\lambda_n = n^{\rho}) \quad & P_{\rho}(t, \theta) &= 1 + 2 \sum_{n=1}^{\infty} e^{-n^{\rho} t} \cos n\theta \quad (t > 0; 0 < \rho < 1); \\
&\text{(c) } (\lambda_n = \log n) \quad & D(t, \theta) &= 1 + 2 \sum_{n=1}^{\infty} n^{-t} \cos n\theta \qquad (t > 0); \\
&\text{(d) } (\lambda_n = n^2) \quad & H(t, \theta) &= \sum_{n=1}^{\infty} e^{-n^2 t} \sin n\theta;
\end{aligned} \tag{2}$$

(a) is the Poisson kernel, (c) the Dirichlet kernel, and (d) the heat equation kernel.

When these functions occur, they usually occur in a subsidiary role, with the result that their properties are developed in an *ad hoc* manner, and only insofar as required at the moment. Since they possess many properties of intrinsic interest, we propose to consider these kernels as objects of independent attention. Many of the results can be developed to include higher dimensional kernels, and as a matter of fact the three dimensional case of the heat equation kernel is treated.

It was shown by Paley [6], in a paper devoted to a theorem on averages due to Hardy and Littlewood, that

$$P(t_1 + t_2, \theta) \leq c_1(P(t_1, \theta) + P(t_2, \theta)), \tag{3}$$

where c_1 is a constant independent of t_1, t_2, and θ. Since, as is easily verified,

$$\frac{1}{2\pi} \int_0^{2\pi} P(t_1, \theta_1 - \theta) P(t_2, \theta - \theta_2)\, d\theta = P(t_1 + t_2, \theta_1 - \theta_2), \tag{4}$$

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