FINITE SUMS AND INTERPOLATION FORMULAS OVER $GF[p^n, x]$

By L. Carlitz

1. **Introduction.** The notation used is that of [2]; in particular polynomials $\varepsilon\ GF[p^n, x]$ will be denoted by $A, B, \cdots, M$. If t is a second indeterminate it is easily proved (§2) that a polynomial $f(t)$ satisfies $f(t + A) = f(t)$ for all A of degree $<m$ if and only if $f(t) = g(\psi_m(t))$, where $g(t)$ is also a polynomial and

$$(1.1) \qquad \psi_m(t) = \prod_{\deg M<m} (t - A) = \sum_{i=0}^{m} (-1)^{m-i} \begin{bmatrix} m \\ i \end{bmatrix} t^{p^{ni}}.$$

We accordingly discuss (§3) the equation

$$(1.2) \qquad \sum_{\deg A<m} h(t + A) = g(\psi_m(t)).$$

In §4 we consider sums of the type $\sum_M h(M)M(t)$ and also give some criteria for the vanishing of

$$\sum_{\deg M=m}' M^r, \qquad \sum_{\deg M<m} M^r \qquad (r > 0).$$

(The notation $\sum'$ indicates that the summation is over primary M only.)

In the remainder of the paper we set up various interpolation formulas (see (5.2), (5.5), (6.1), (6.2), (6.12), (7.1), (8.3)). Of the applications, we mention the theorem that *a polynomial $g(t)$ of degree $<p^{nm}$ is integral-valued if and only if $g(M)$ is integral for all M of degree $<m$.* For other applications we cite Theorems 8.1 and 8.2.

2. Some preliminary theorems.

Theorem 2.1. *A polynomial $f(t)$ with arbitrary coefficients satisfies*

$$(2.1) \qquad f(t + A) = f(A)$$

for all $A\ \varepsilon\ GF[p^n, x]$ of degree $<m$ if and only if $f(t) = g(\psi_m(t))$, where $g(t)$ is a polynomial.

Proof. By (1.1), $\psi_m(t + A) = \psi_m(t)$ for $\deg A < m$ and therefore $g(\psi_m(t + A)) = g(\psi_m(t))$.

To prove the converse put

$$(2.2) \qquad f(t) = h_0(t) + h_1(t)\psi_m(t) + h_2(t)\psi_m^2(t) + \cdots,$$

where $\deg h_i(t) < p^{nm}$. If we replace t by $t + A$ and equate coefficients we get $h_i(t + A) = h_i(t)$. Hence it is only necessary to show that if $h(t + A) = h(t)$, $\deg h(t) < p^{nm}$, $\deg A < m$, then $h(t)$ is constant. Indeed if $h(t_0) = 0$ then

Received August 30, 1948.