BASES IN BANACH SPACES

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Introduction. This paper investigates the relationship of a basis and the structure of the space. The assumption of the existence of an absolute basis in the space E and the additional requirement of separability for the conjugate space E^* implies that E^* is weakly complete. Other criteria for the existence of a basis in the space are discussed. The relationship of weak, absolute, absolutely convergent bases are analyzed. Some unsolved questions are posed at the close of the paper.

Preliminaries. In this section we introduce the concepts of a basis and an absolute basis.

Definition 1. A sequence of elements x_n in E (separable Banach space) form a basis for E, if for every x in E there exists a unique set of real numbers a_n such that $x = \sum_1^\infty a_n x_n$, that is $\| x - \sum_1^m a_n x_n \| \to 0$.

It is well known that the set of real numbers a_n defines linear functionals over E. We denote them by $a_n = f_n(x)$ (see [1; 111]). The functionals f_n are biorthogonal with respect to x_n and we have that $f_n(x_m) = \delta_{nm}$ where $\delta_{nm} = 0$ $(n \neq m)$, $= 1$ $(n = m)$. Whenever x_m forms a basis, then x_n is total over E^*; that is, if $f(x_n) = 0$ for every n then $f = 0$ (E^* denotes the conjugate space of E). In c_0 (space of sequences convergent to 0) and l^p (space of sequences whose series is to the p-th power convergent) an example of a basis is $x_n = \{\delta_{nm}\}_m$. The Haar orthogonal system forms a basis for L^p $(p \geq 1)$. If x_n constitutes a basis for E and f_n denotes its biorthogonal sequence, then $\sum f_n f(x_n)$ converges for every f in the linear manifold spanned by $\{f_n\}$. More generally, if $\| \sum_{n=1}^m f_n f(x_n) \| \leq C$ for every m and every f with $\| f \| \leq 1$, then the same conclusion holds. The inequalities $\| \sum_{n=1}^m f_n f(x_n) \| \leq C$ and $\| \sum_{n=1}^k f_n(x) x_n \| \leq C$ are equivalent. These facts are discussed in Banach's book [1; 111, 112].

Definition 2. A basis x_m for E is an absolute basis if whenever $\sum a_n x_n$ converges, then $\sum_{L=1}^\infty a_{n_L} x_{n_L}$ converges for every subsequence of indices $\{n_L\}$. In other words, $\sum a_n x_n$ converges unconditionally.

Associated with the concept of an absolute basis, the notion of projection operations will play an important role.

A projection P of E on M $(M \subset E)$ is a linear bounded operator such that $P^2 = P$ and $P(E) = M$. A closed linear set (manifold) M in E is said to have a complement in E if there exists a closed linear set N such that $M \cap N = 0$ and $E = M \oplus N$ (direct sum). Manifolds M possessing complements are equivalent to the property of possessing a projection on M (see [7; 138]).

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